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Addendum to: Event-Based State Estimation with Variance-Based Triggering

Technical Report

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Abstract

This report states the proofs of Propositions 1 to 6 and Corollary 2 in [1]. All references (equations, propositions, etc.) continue those in [1].

1 Proof of Proposition 1

Proof. Recall the definitions of h and g in (23) and (24). Notice that

$$p_1 = h(\bar{p}+\delta) = g(\bar{p}+\delta) < \bar{p}+\delta < a^2(\bar{p}+\delta)+1 = p_2. \tag{44}$$

We first show that

$$h(p_1) > h(p_2), \tag{45}$$

which is useful later. Let $\tilde{p} := \bar{p}+\delta$. With (44),

$$\begin{aligned} h(p_1) &= a^2 p_1 + 1 = a^2 g(\tilde{p}) + 1 \\ &= a^4\tilde{p} + a^2 + 1 - \frac{a^4c^2\tilde{p}^2}{c^2\tilde{p}+1} \qquad \text{and} \qquad (46)\\ h(p_2) &= g(p_2) = g(a^2\tilde{p}+1) \\ &= a^4\tilde{p} + a^2 + 1 - \frac{a^2c^2(a^2\tilde{p}+1)^2}{c^2(a^2\tilde{p}+1)+1}. \qquad (47) \end{aligned}$$

Hence,

$$\begin{aligned} h(p_1) - h(p_2) &= -\frac{a^4c^2\tilde{p}^2}{c^2\tilde{p}+1} + \frac{a^2c^2(a^2\tilde{p}+1)^2}{c^2(a^2\tilde{p}+1)+1} \\ &= \frac{-a^4c^2\tilde{p}^2(c^2(a^2\tilde{p}+1)+1) + a^2c^2(a^2\tilde{p}+1)^2(c^2\tilde{p}+1)}{(c^2\tilde{p}+1)(c^2(a^2\tilde{p}+1)+1)} \\ &= \frac{a^4c^4\tilde{p}^2 + a^4c^2(a^2-1)\tilde{p}^2 + a^2c^4\tilde{p} + 2a^4c^2\tilde{p} + a^2c^2}{(c^2\tilde{p}+1)(a^2c^2\tilde{p}+c^2+1)}. \end{aligned} \tag{48}$$

For the assumed parameter values ($|a| > 1$, $c \neq 0$), the numerator and denominator are strictly greater than 0. Hence, $h(p_1) - h(p_2) > 0$, from which (45) follows.

Next, we prove the statements of the proposition.
(iv), (v): For $p \in [p_1, \bar{p}+\delta)$,

$$h(p) = a^2 p + 1, \tag{49}$$

which is a continuous and strictly monotonic increasing function of p because $|a| > 1$. Furthermore, h is differentiable for $p \in (p_1, \bar{p}+\delta)$ with $h'(p) = a^2$. For $p \in [\bar{p}+\delta, p_2]$,

$$h(p) = g(p) = a^2p + 1 - \frac{a^2c^2p^2}{c^2p+1} = \frac{a^2p + c^2p + 1}{c^2p+1}, \tag{50}$$

which is continuous since the denominator has no zero for positive p. Since, for $p \in (0, \infty)$, g is differentiable with

$$g'(p) = \frac{a^2}{(c^2p+1)^2} > 0, \tag{51}$$

h is strictly monotonic increasing on $[\bar{p}+\delta, p_2]$ and differentiable on $(\bar{p}+\delta, p_2)$.

(iii): h is injective on each of the intervals $[p_1, \bar{p}+\delta)$ and $[\bar{p}+\delta, p_2)$ separately by continuity and monotonicity (iv). Furthermore, by strict monotonicity,

$$\begin{aligned} h([p_1, \bar{p}+\delta)) &= [h(p_1), \lim_{p \nearrow \bar{p}+\delta} h(p)) \\ &= [h(p_1), a^2(\bar{p}+\delta)+1) = [h(p_1), p_2), \quad \text{and} \qquad (52) \\ h([\bar{p}+\delta, p_2)) &= [h(\bar{p}+\delta), \lim_{p \nearrow p_2} h(p)) = [p_1, h(p_2)), \qquad (53) \end{aligned}$$

where $p \nearrow \bar{p}+\delta$ denotes the left-sided limit, i.e. p approaches $\bar{p}+\delta$ from below. From (45), $[p_1, h(p_2)) \cap [h(p_1), p_2) = \emptyset$. Therefore, h is injective on $[p_1, p_2)$.

(i): Follows from (52), (53), and (45).

(ii): Consider three cases for $p \in [0, \infty)$:

- $p \in [0, p_1)$. We first show that the sequence $h^k(p), k \geq 0$ eventually is greater than p_1. For $h^k(p) \in [0, p_1)$, $h^{k+1}(p) = a^2 h^k(p) + 1 > a^2 h^k(p)$. Hence, for $p, h(p), \ldots, h^{k-1}(p) \in [0, p_1)$, $h^k(p) > a^{2k} p$. But, since $\lim_{k\to\infty} a^{2k} p = \infty$, there exists an $m \in \mathbb{N}$ such that

$$h^{m-1}(p) \in [0, p_1) \quad \text{and} \quad h^m(p) \in [p_1, \infty). \qquad (54)$$

  Next, notice that

$$h([0, p_1)) = [h(0), h(p_1)) = [1, h(p_1)) \subseteq [1, p_2) \qquad (55)$$

  because $h(p_1) < p_2$ by (i). Since $h^{m-1}(p) \in [0, p_1)$, it follows that $h^m(p) = h(h^{m-1}(p)) \in [1, p_2)$. Together with (54), this implies that $h^m(p) \in [p_1, \infty) \cap [1, p_2) = [p_1, p_2)$.

- $p \in [p_1, p_2)$. Take $m = 1$ and the claim follows from (i).

- $p \in [p_2, \infty)$. Since $h(p) = g(p)$, the sequence $h^k(p) = g^k(p)$ evolves as for the full communication Kalman filter. By the convergence properties of the full communication Kalman filter, [2], $\lim_{k\to\infty} g^k(p) = \bar{p}$ and, by (44), $\bar{p} < \bar{p}+\delta < p_2$. Hence, there exists an $m \in \mathbb{N}$ such that

$$h^{m-1}(p) \in [p_2, \infty) \quad \text{and} \quad h^m(p) \in [0, p_2). \qquad (56)$$

  Since

$$h([p_2, \infty)) \subseteq [h(p_2), \infty) = [h(a^2(\bar{p}+\delta)+1), \infty) \underset{\text{(iv)}}{\subseteq} [h(\bar{p}+\delta), \infty) = [p_1, \infty),$$

  $h^m(p) = h(h^{m-1}(p)) \in [p_1, \infty)$. Therefore, $h^m(p) \in [0, p_2) \cap [p_1, \infty) = [p_1, p_2)$.

□

2 Proof of Proposition 2

Proof. (i): By Assumption 1, the sequence $\{d_1, d_2, \ldots\}$ defined by Algorithm 1 is finite and equal to $\mathcal{D}_{N-1}$. Therefore, $d_i \in \operatorname{dom}(h^{-1})$ for all $i < N-1$ and $d_{N-1} \notin \operatorname{dom}(h^{-1})$. From $\operatorname{dom}(h^{-1}) = [p_1, h(p_2)) \cup [h(p_1), p_2)$ (see (28)), it follows directly that $d_i \notin [h(p_2), h(p_1))$ for all $i < N-1$. Since h^{-1} maps to $[p_1, p_2)$ (see (28)), we have $d_{N-1} = h^{-1}(d_{N-2}) \in [p_1, p_2)$. Together with $d_{N-1} \notin \operatorname{dom}(h^{-1})$, this implies that $d_{N-1} \in [p_1, p_2) \setminus \big([p_1, h(p_2)) \cup [h(p_1), p_2)\big) = [h(p_2), h(p_1))$.

(ii): First, we prove by induction that h^i is continuous on $[p_1, p_2) \setminus \mathcal{D}_i$ for all $i \leq N-1$. From Proposition 1, (iv), it follows that the statement is true for $i = 1$. Assume the statement holds for $i \in \{1, \ldots, N-2\}$ (induction assumption (IA)). Consider

$$h^{i+1}(p) = h(h^i(p)), \quad p \in [p_1, p_2). \tag{57}$$

If h^i is continuous at p and h is continuous at $h^i(p)$, then the composition h^{i+1} is continuous at p, [3]. Hence, h^{i+1} is continuous on $[p_1, p_2)$ except for the points $\mathcal{D}_i$ (discontinuities of h^i by IA) and the point $\tilde{p}$ with $h^i(\tilde{p}) = d_1$ (d_1 is the discontinuity of h). But $h^i(\tilde{p}) = d_1 \Leftrightarrow \tilde{p} = h^{-i}(d_1) = d_{i+1}$ (since $i \leq N-2$, the i times application of the inverse map is defined). Therefore, h^{i+1} is continuous on $[p_1, p_2) \setminus (\mathcal{D}_i \cup \{d_{i+1}\}) = [p_1, p_2) \setminus \mathcal{D}_{i+1}$.

Next, we prove that h^N is continuous on $[p_1, p_2) \setminus \mathcal{D}_{N-1}$. For this, consider

$$h^N(p) = h(h^{N-1}(p)), \quad p \in [p_1, p_2). \tag{58}$$

By the same argument as above, h^N is continuous on $[p_1, p_2)$ except for the points $\mathcal{D}_{N-1}$ and the point $\tilde{p}$ with $h^{N-1}(\tilde{p}) = d_1 \Leftrightarrow h(\tilde{p}) = h^{N-1-(N-2)}(\tilde{p}) = h^{-(N-2)}(d_1) = d_{N-1}$. But a point $\tilde{p}$ with $h(\tilde{p}) = d_{N-1}$ does not exist in $[p_1, p_2)$ since $d_{N-1} \in [h(p_2), h(p_1))$ (by (i)), which is not in the domain of h^{-1} (see (28)). Therefore, h^N is continuous on $[p_1, p_2) \setminus \mathcal{D}_{N-1}$.

(iii): Proof by contradiction. Assume there exist $d_i, d_j \in \mathcal{D}_{N-1}$ with $i \neq j$ and $d_i = d_j$. Assume w.l.o.g. $j > i$ and let $M := j - i \leq N - 2$. Then, from Algorithm 1,

$$d_i = d_j = h^{-1}(d_{j-1}) = h^{-2}(d_{j-2}) = \cdots = h^{-M}(d_i). \tag{59}$$

It follows that, for all $\ell \in \{0, \ldots, M-1\}$,

$$d_{i+\ell} = h^{-\ell}(d_i) = h^{-\ell}(h^{-M}(d_i)) = h^{-M}(h^{-\ell}(d_i)) = h^{-M}(d_{i+\ell}), \tag{60}$$

that is, the sequence $\{d_i, d_{i+1}, \ldots\}$ is periodic with period M. But then, for all $\ell \in \{0, \ldots, M-1\}$ and $m \in \mathbb{N}$,

$$d_{i+\ell+mM} = h^{-mM}(d_{i+\ell}) = d_{i+\ell}\,, \tag{61}$$

that is, Algorithm 1 never terminates, which contradicts with Assumption 1. □

3 Proof of Proposition 3

Proof. The intervals are disjoint by construction.

Because of Proposition 2, (iii), the intervals in (36) are not empty. Since for all $d_i \in \mathcal{D}_{N-1}$, $d_i \in [p_1, p_2)$, which implies $d_{\bar{i}} < p_2$; and, therefore, interval $I_{\bar{i}}$ in (37) is not empty. To see that I_N in (38) is not empty, consider the case where it is and show that this leads to a contradiction. From $[p_1, d_{\underline{i}}) = \emptyset$ it follows that $p_1 = d_{\underline{i}}$ ($p_1 > d_{\underline{i}}$ is not possible since $d_{\underline{i}} \in [p_1, p_2)$). From $d_{\underline{i}} = p_1 \in \mathrm{dom}(h^{-1})$, it follows that $d_{\underline{i}+1}$ is defined by Algorithm 1: $d_{\underline{i}+1} = h^{-1}(d_{\underline{i}}) = h^{-1}(p_1) = h^{-1}(h(\bar{p} + \delta)) = \bar{p} + \delta = d_1$. But $d_{\underline{i}+1} = d_1$ with $\underline{i} \geq 1$ contradicts with Proposition 2, (iii). □

4 Proof of Proposition 4

We first state two lemmas and one corollary that are used in the proof of Proposition 4 at the end of this section.

Lemma 1. *Let $\mathcal{I} = \{I_1, I_2, \ldots, I_N\}$ be a collection of nonempty, mutually disjoint intervals $I_i := [a_i, b_i)$ (or $I_i := (a_i, b_i)$) for $a_i, b_i \in \mathbb{R}$. A unique representation of $\mathcal{I}$ is given by the sets*

$$\mathcal{L} = \{a_1, a_2, \ldots, a_N\} \quad \text{and} \tag{62}$$

$$\mathcal{U} = \{b_1, b_2, \ldots, b_N\}, \tag{63}$$

of all lower and upper bounds, respectively, in the following sense: the collection $\bar{\mathcal{I}}$ of intervals defined by

$$\bar{\mathcal{I}} := \{\bar{I}_1, \bar{I}_2, \ldots, \bar{I}_N\}, \quad \bar{I}_i := [\alpha_i, \beta_i) \text{ (or } \bar{I}_i := (\alpha_i, \beta_i)\text{)}, \, \alpha_i \in \mathcal{L}, \, \beta_i \in \mathcal{U}, \tag{64}$$

such that, for all i, j with $1 \le i \le N$, $1 \le j \le N$,

$$\bar{I}_i \neq \emptyset, \quad \text{and} \quad \bar{I}_i \cap \bar{I}_j = \emptyset, \tag{65}$$

exists and it is unique, and $\bar{\mathcal{I}} = \mathcal{I}$.

This lemma is useful, since it allows to work with the (unordered) set of interval bounds $\mathcal{L}$ and $\mathcal{U}$ instead of the actual intervals. The unique relationship between the bounds (which lower bound belongs to which upper bound) essentially follows from all intervals being disjoint and nonempty.

Proof. [3] Since, for all $i \le N$, $I_i \in \mathcal{I}$ is nonempty, $a_i < b_i$. Since the intervals $\mathcal{I}$ are mutually disjoint, there exists a permutation of indices $\tilde{\Pi} : \{1, \ldots, N\} \to \{1, \ldots, N\}$ such that

$$a_{\tilde{\Pi}(1)} < b_{\tilde{\Pi}(1)} \le a_{\tilde{\Pi}(2)} < b_{\tilde{\Pi}(2)} \le \cdots \le a_{\tilde{\Pi}(N)} < b_{\tilde{\Pi}(N)}. \tag{66}$$

Assume w.l.o.g. (by renaming of the intervals in $\mathcal{I}$) that

$$a_1 < b_1 \le a_2 < b_2 \le \cdots \le a_N < b_N. \tag{67}$$

Notice that the choice $\bar{\mathcal{I}}_1 = \{\bar{I}_1, \bar{I}_2, \ldots, \bar{I}_N\}$ with $\bar{I}_i = [a_i, b_i)$ ($\bar{I}_i = (a_i, b_i)$) satisfies (64)–(65), and $\bar{\mathcal{I}}_1 = \mathcal{I}$. Hence, a collection of intervals $\bar{\mathcal{I}}$ according to (64)–(65) exists. It remains to show that $\bar{\mathcal{I}}_1$ is unique; that is, $\bar{\mathcal{I}}_1$ is the only collection of intervals satisfying (64)–(65).

First notice that, for any $a_i \in \mathcal{L}$, there is exactly one interval in $\bar{\mathcal{I}}$ that has a_i as a lower bound. We will show this by contradiction.

- Assume there is more than one interval with a_i as a lower bound; that is, there are $[a_i, b_j)$, $[a_i, b_\ell) \in \bar{\mathcal{I}}$ ($(a_i, b_j), (a_i, b_\ell) \in \bar{\mathcal{I}}$) with $b_j, b_\ell \in \mathcal{U}$ and $b_j > a_i$, $b_\ell > a_i$ (otherwise the intervals would be empty, which contradicts with (65)). But then,

$$[a_i, b_j) \cap [a_i, b_\ell) = [a_i, \min(b_j, b_\ell)) \neq \emptyset, \tag{68}$$
$$\left(\, (a_i, b_j) \cap (a_i, b_\ell) = (a_i, \min(b_j, b_\ell)) \neq \emptyset \, \right)$$

  which contradicts with (65).

[3] We present the proof simultaneously for the case of left-closed, right-open intervals $\bar{I}_i = [\alpha_i, \beta_i)$ and for the case of open intervals $\bar{I}_i = (\alpha_i, \beta_i)$. Where required, we distinguish the two cases in the text by writing the latter case in parentheses.

- Assume there is no interval in $\bar{\mathcal{I}}$ that has a_i as a lower bound. Then, there can only be $N-1$ intervals in total, since it follows from the previous discussion that each of the remaining $a_j \in \mathcal{L} \setminus \{a_i\}$ can be chosen at most once as a lower bound. This contradicts with (64) (the collection $\bar{\mathcal{I}}$ having N elements).

Similarly, note that for any $b_i \in \mathcal{U}$, there is exactly one interval in $\bar{\mathcal{I}}$ that has b_i as an upper bound.

- Assume there is more than one interval with b_i as an upper bound; that is, there are $[a_j, b_i), [a_\ell, b_i) \in \bar{\mathcal{I}}$ ($(a_j, b_i), (a_\ell, b_i) \in \bar{\mathcal{I}}$) with $a_j, a_\ell \in \mathcal{L}$ and $a_j < b_i$, $a_\ell < b_i$ (otherwise the intervals would be empty). But then,

$$[a_j, b_i) \cap [a_\ell, b_i) = [\max(a_j, a_\ell),\, b_i) \neq \emptyset, \qquad (69)$$
$$(\ (a_j, b_i) \cap (a_\ell, b_i) = (\max(a_j, a_\ell),\, b_i) \neq \emptyset\ )$$

  which contradicts with (65).

- Assume there is no interval in $\bar{\mathcal{I}}$ that has b_i as an upper bound. Then, there can only be $N-1$ intervals in total, since each of the remaining $b_j \in \mathcal{U} \setminus \{b_i\}$ can be chosen at most once as an upper bound. This contradicts with (64).

Now, take any $a_i \in \mathcal{L}$. From the discussion above, it follows that there is an interval $[a_i, b_j) \in \bar{\mathcal{I}}$ ($(a_i, b_j) \in \bar{\mathcal{I}}$), $b_j \in \mathcal{U}$. We prove by contradiction that this implies $b_j = b_i$, and, hence, that $\bar{\mathcal{I}}_1 = \bar{\mathcal{I}}$ is unique.

Let $b_i \in \mathcal{U}$ and assume $b_j \neq b_i$. Then, from the above discussion, there exists also an interval $[a_\ell, b_i) \in \bar{\mathcal{I}}$ ($(a_\ell, b_i) \in \bar{\mathcal{I}}$), $a_\ell \in \mathcal{L}$. For $[a_i, b_j)$ ((a_i, b_j)) to be nonempty, it follows that

$$a_i < b_j \quad \underset{(67)}{\Rightarrow} \quad b_i \leq b_j\,; \qquad (70)$$

and, for $[a_\ell, b_i)$ ((a_ℓ, b_i)) to be nonempty,

$$a_\ell < b_i \quad \underset{(67)}{\Rightarrow} \quad a_\ell \leq a_i. \qquad (71)$$

But then,

$$[a_i, b_j) \cap [a_\ell, b_i) = [a_i,\, b_i) \neq \emptyset, \qquad (72)$$
$$(\ (a_i, b_j) \cap (a_\ell, b_i) = (a_i,\, b_i) \neq \emptyset\ ),$$

which contradicts (65). □

Corollary 3. *Let $\mathcal{I}_1$, $\mathcal{I}_2$ be two collections of nonempty and mutually disjoint intervals. Let $\mathcal{L}_1$ and $\mathcal{U}_1$ be the sets of lower and upper bounds, respectively, of $\mathcal{I}_1$; and let $\mathcal{L}_2$ and $\mathcal{U}_2$ be the sets of lower and upper bounds, respectively, of $\mathcal{I}_2$. If $\mathcal{L}_1 = \mathcal{L}_2$ and $\mathcal{U}_1 = \mathcal{U}_2$, then $\mathcal{I}_1 = \mathcal{I}_2$.*

Proof. Let $\bar{\mathcal{I}}_1$ be constructed from $\mathcal{L}_1$ and $\mathcal{U}_1$ according to (64)–(65). Then $\bar{\mathcal{I}}_1 = \mathcal{I}_1$ by Lemma 1. Furthermore, let $\bar{\mathcal{I}}_2$ be constructed from $\mathcal{L}_2$ and $\mathcal{U}_2$ according to (64)–(65). Then $\bar{\mathcal{I}}_2 = \mathcal{I}_2$ by Lemma 1.

Since $\bar{\mathcal{I}}_1$ and $\bar{\mathcal{I}}_2$ are unique, $\mathcal{L}_1 = \mathcal{L}_2$ and $\mathcal{U}_1 = \mathcal{U}_2$ implies $\bar{\mathcal{I}}_1 = \bar{\mathcal{I}}_2$, and, therefore, $\mathcal{I}_1 = \mathcal{I}_2$. □

We give two definitions that are used in the following Lemma and in subsequent sections.

Definition 3. *Let f be a function, and let $\mathcal{I}_1$, $\mathcal{I}_2$ be collections of intervals. We write*

$$\mathcal{I}_1 \xrightarrow{f} \mathcal{I}_2 \tag{73}$$

to denote

$$\forall I_1 \in \mathcal{I}_1, \exists I_2 \in \mathcal{I}_2 \;:\; f(I_1) \subseteq I_2. \tag{74}$$

Definition 4. *Define the binary operator '$-_N$' as follows: for $\alpha, \beta \in \mathbb{Z}$ and $N \in \mathbb{N}$,*

$$\alpha -_N \beta = \begin{cases} \operatorname{mod}(\alpha - \beta, N) & \textit{if } \operatorname{mod}(\alpha - \beta, N) > 0 \\ N & \textit{if } \operatorname{mod}(\alpha - \beta, N) = 0, \end{cases} \tag{75}$$

where $\operatorname{mod}(\gamma, N) \in \{0, \ldots, N-1\}$ is the (positive) remainder of $\gamma \in \mathbb{Z}$ divided by N. Hence, '$-_N$' is the subtraction with subsequent modulo N operation, except that a resulting 0 is replaced by N.

Lemma 2. *Consider the collection $\mathcal{I} = \{I_1, I_2, \ldots, I_N\}$ of intervals I_i defined by (36)–(38); and let $\mathcal{I}_{\text{int}} := \{\operatorname{int}(I_1), \ldots, \operatorname{int}(I_N)\}$. The following statements hold:*

(i) $\mathcal{I} \xrightarrow{h} \mathcal{I}$.

(ii) $\mathcal{I}_{\text{int}} \xrightarrow{h} \mathcal{I}_{\text{int}}$.

(iii) $I_{\bar{i} -_N 1} = \begin{cases} [d_{\bar{i}-1}, d_{N-1}) & \bar{i} > 1 \\ [p_1, d_{N-1}) & \bar{i} = 1. \end{cases}$

(iv) $\operatorname{int}(I_{N-1}) = \begin{cases} (d_{N-1}, d_{\underline{i}-1}) & \underline{i} > 1 \\ (d_{N-1}, p_2) & \underline{i} = 1. \end{cases}$

Statements (i) and (ii) are used in the proof of Proposition 4 later in this section. Statements (iii) and (iv) are used in Sec. 5.

Proof. (i), (ii)[4]: By Proposition 3, the intervals

$$\begin{aligned} \mathcal{I} &= \big\{I_1, I_2, \ldots, I_N\big\} \\ &= \big\{[p_1, d_{\Pi(1)}),\, [d_{\Pi(1)}, d_{\Pi(2)}),\, \ldots,\, [d_{\Pi(N-1)}, p_2)\big\} \end{aligned} \tag{76}$$

are mutually disjoint and nonempty. Therefore, also the intervals

$$\begin{aligned} \mathcal{I}_{\text{int}} &= \big\{\operatorname{int}(I_1), \operatorname{int}(I_2), \ldots, \operatorname{int}(I_N)\big\} \\ &= \big\{(p_1, d_{\Pi(1)}),\, (d_{\Pi(1)}, d_{\Pi(2)}),\, \ldots,\, (d_{\Pi(N-1)}, p_2)\big\} \end{aligned} \tag{77}$$

are mutually disjoint and nonempty. Hence, by Lemma 1, $\mathcal{I}$ ($\mathcal{I}_{\text{int}}$) is uniquely represented by

$$\mathcal{L} = \big\{p_1, d_{\Pi(1)}, \ldots, d_{\Pi(N-1)}\big\} = \big\{p_1, d_1, \ldots, d_{N-1}\big\}, \tag{78}$$

$$\mathcal{U} = \big\{d_{\Pi(1)}, \ldots, d_{\Pi(N-1)}, p_2\big\} = \big\{p_2, d_1, \ldots, d_{N-1}\big\} \tag{79}$$

[4]We present the proof simultaneously for (i) and (ii). Where required, we distinguish the two cases in the text by writing the latter case in parentheses.

and (64)–(65) (note that (64) is a different definition for $\mathcal{I}$ and $\mathcal{I}_{\text{int}}$).

Define

$$\mathcal{I}_h := \big\{h\big([p_1, d_{\Pi(1)})\big),\ h\big([d_{\Pi(1)}, d_{\Pi(2)})\big),\ \ldots,\ h\big([d_{\Pi(N-1)}, p_2)\big)\big\} \tag{80}$$

$$\Big(\ \mathcal{I}_{\text{int},h} := \big\{h\big((p_1, d_{\Pi(1)})\big),\ h\big((d_{\Pi(1)}, d_{\Pi(2)})\big),\ \ldots,\ h\big((d_{\Pi(N-1)}, p_2)\big)\big\}\ \Big), \tag{81}$$

the collection of images of h on $\mathcal{I}$ ($\mathcal{I}_{\text{int}}$). Hence, by definition,

$$\mathcal{I} \xrightarrow{h} \mathcal{I}_h \tag{82}$$

$$\Big(\ \mathcal{I}_{\text{int}} \xrightarrow{h} \mathcal{I}_{\text{int},h}\ \Big). \tag{83}$$

Since, by Proposition 1, (iv), h is continuous and strictly monotonic increasing on each $I_i \in \mathcal{I}$ ($I_i \in \mathcal{I}_{\text{int}}$), the sets of lower and upper bounds of $\mathcal{I}_h$ ($\mathcal{I}_{\text{int},h}$) are given by

$$\begin{aligned}\mathcal{L}_h := \big\{h(a) \mid a \in \mathcal{L}\big\} &= \big\{h(p_1), h(d_1), h(d_2), \ldots, h(d_{N-1})\big\} \\ &= \big\{h(p_1), p_1, d_1, \ldots, d_{N-2}\big\},\end{aligned} \tag{84}$$

$$\begin{aligned}\mathcal{R}_h := \big\{\lim_{p\nearrow b} h(p) \mid b \in \mathcal{U}\big\} &= \big\{h(p_2), \lim_{p\nearrow d_1} h(p), h(d_2), \ldots, h(d_{N-1})\big\} \\ &= \big\{h(p_2), p_2, d_1, \ldots, d_{N-2}\big\},\end{aligned} \tag{85}$$

where we used the facts that h is continuous from the right at all $a \in \mathcal{L}$ and continuous from the left at all $b \in \mathcal{U} \setminus \{d_1\}$; and that

$$h(d_1) = h(\bar{p} + \delta) = p_1 \qquad \text{(by definition of } p_1\text{)}, \tag{86}$$

$$h(d_i) = d_{i-1}, \quad \forall i \in \{2, \ldots, N-1\} \qquad (d_i = h^{-1}(d_{i-1}) \text{ from Alg. 1}), \tag{87}$$

$$\lim_{p\nearrow d_1} h(p) = \lim_{p\nearrow \bar{p}+\delta} h(p) = a^2(\bar{p} + \delta) + 1 = p_2 \qquad \text{(by definition of } p_2\text{)}. \tag{88}$$

Since h is injective (Proposition 1, (iii)), $h(I_1 \cap I_2) = h(I_1) \cap h(I_2)$ holds for any $I_1, I_2 \subseteq [p_1, p_2)$, [4]. From this and the intervals $\mathcal{I}$ ($\mathcal{I}_{\text{int}}$) being disjoint, it follows that the mapped intervals $\mathcal{I}_h$ ($\mathcal{I}_{\text{int},h}$) are also disjoint. Furthermore, since h is not constant on any interval $I \in \mathcal{I}$ (it is strictly monotonic increasing by Proposition 1, (iv)), the intervals $\mathcal{I}_h$ ($\mathcal{I}_{\text{int},h}$) are all nonempty. Hence, by Lemma 1, $\mathcal{I}_h$ ($\mathcal{I}_{\text{int},h}$) is uniquely represented by $\mathcal{L}_h$ and $\mathcal{U}_h$.

Notice that $\mathcal{L}_h$ and $\mathcal{U}_h$ have the same elements as $\mathcal{L}$ and $\mathcal{U}$ except for $h(p_1)$ and $h(p_2)$ in $\mathcal{L}_h$ and $\mathcal{U}_h$, and d_{N-1} in $\mathcal{L}$ and $\mathcal{U}$. We show next that the intervals $\mathcal{I}_h$ ($\mathcal{I}_{\text{int},h}$) are contained in $\mathcal{I}$ ($\mathcal{I}_{\text{int}}$).

To see this, notice first that the elements of $\mathcal{L}_h \cup \mathcal{U}_h \cup \mathcal{L} \cup \mathcal{U} = \{p_1, p_2, h(p_1), h(p_2), d_1, \ldots, d_{N-1}\}$ have the following order relation:

$$p_1 \le \underbrace{\cdots\cdots}_{\text{other } d_i\text{'s}} < h(p_2) \le d_{N-1} < h(p_1) \le \underbrace{\cdots\cdots}_{\text{other } d_i\text{'s}} < p_2, \tag{89}$$

because

$$p_1 < h(p_2) \qquad \text{(by (44) and Proposition 1, (iv))},$$

$$h(p_1) < p_2 \qquad \text{(by Proposition 1, (i))},$$

$$h(p_2) \le d_{N-1} < h(p_1) \qquad \text{(by Proposition 2, (i))},$$

$$d_i \in [p_1, h(p_2)) \cup [h(p_1), p_2),\ \forall i \in \{1, \ldots, N-2\} \qquad \text{(by Proposition 2, (i))}.$$

Therefore, the upper bound of $[*, h(p_2)) \in \mathcal{I}_h$ ($(*, h(p_2)) \in \mathcal{I}_{\text{int},h}$) can be changed to d_{N-1}, and the lower bound of $[h(p_1), *) \in \mathcal{I}_h$ ($(h(p_1), *) \in \mathcal{I}_{\text{int},h}$) to d_{N-1}, without affecting the mutual disjointness and non-emptiness of the intervals. This is illustrated in Fig. 10.

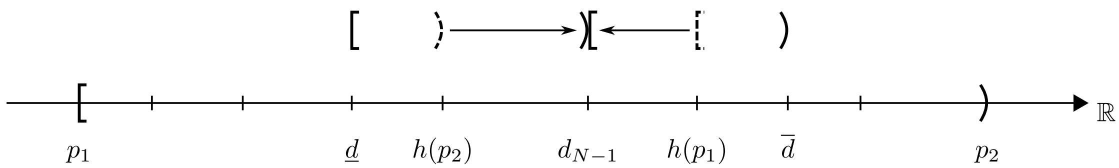

Figure 10: Illustration of the enlargement of the intervals $[\underline{d}, h(p_2))$ and $[h(p_1), \overline{d})$ to $[\underline{d}, d_{N-1})$ and $[d_{N-1}, \overline{d})$. The points unspecified are elements from $\{d_1, \ldots, d_{N-2}\}$. All intervals remain nonempty and mutually disjoint.

Let $\underline{d}$ be the lower bound of $[*, h(p_2)) \in \mathcal{I}_h$ ($(*, h(p_2)) \in \mathcal{I}_{\text{int},h}$), and let $\overline{d}$ be the upper bound of $[h(p_1), *) \in \mathcal{I}_h$ ($(h(p_1), *) \in \mathcal{I}_{\text{int},h}$) (cf. Fig. 10). Note that $\underline{d}$ and $\overline{d}$ are unique since by the disjointness and nonemptiness of the intervals, there is exactly one interval with $h(p_2)$ as an upper bound, and there is exactly one interval with $h(p_1)$ as a lower bound. Then, define

$$\begin{aligned}\tilde{\mathcal{I}}_h := &\big\{ I \in \mathcal{I}_h \mid I \neq [\underline{d}, h(p_2)) \text{ and } I \neq [h(p_1), \overline{d}) \big\} \\ &\cup \big\{ [\underline{d}, d_{N-1}),\ [d_{N-1}, \overline{d}) \big\},\end{aligned} \tag{90}$$

that is, $\tilde{\mathcal{I}}_h$ has the same elements as $\mathcal{I}_h$ except for the replacements $[\underline{d}, h(p_2)) \to [\underline{d}, d_{N-1})$ and $[h(p_1), \overline{d}) \to [d_{N-1}, \overline{d})$. Similarly, define

$$\begin{aligned}\tilde{\mathcal{I}}_{\text{int},h} := &\big\{ I \in \mathcal{I}_{\text{int},h} \mid I \neq (\underline{d}, h(p_2)) \text{ and } I \neq (h(p_1), \overline{d}) \big\} \\ &\cup \big\{ (\underline{d}, d_{N-1}),\ (d_{N-1}, \overline{d}) \big\}.\end{aligned} \tag{91}$$

Since $[\underline{d}, h(p_2)) \subseteq [\underline{d}, d_{N-1})$ ($(\underline{d}, h(p_2)) \subseteq (\underline{d}, d_{N-1})$) and $[h(p_1), \overline{d}) \subseteq [d_{N-1}, \overline{d})$ ($(h(p_1), \overline{d}) \subseteq (d_{N-1}, \overline{d})$), it follows from (82) and (83) that

$$\mathcal{I} \xrightarrow{h} \tilde{\mathcal{I}}_h \tag{92}$$

$$\Big(\mathcal{I}_{\text{int}} \xrightarrow{h} \tilde{\mathcal{I}}_{\text{int},h} \Big). \tag{93}$$

The lower and upper bounds of $\tilde{\mathcal{I}}_h$ ($\tilde{\mathcal{I}}_{\text{int},h}$) are given by

$$\tilde{\mathcal{L}}_h := \big\{ d_{N-1}, p_1, d_1, \ldots, d_{N-2} \big\}, \tag{94}$$

$$\tilde{\mathcal{U}}_h := \big\{ d_{N-1}, p_2, d_1, \ldots, d_{N-2} \big\}. \tag{95}$$

Since the intervals $\tilde{\mathcal{I}}_h$ ($\tilde{\mathcal{I}}_{\text{int},h}$) are nonempty and mutually disjoint, and $\tilde{\mathcal{L}}_h = \mathcal{L}$ and $\tilde{\mathcal{U}}_h = \mathcal{U}$, it follows from Corollary 3 that $\tilde{\mathcal{I}}_h = \mathcal{I}$ ($\tilde{\mathcal{I}}_{\text{int},h} = \mathcal{I}_{\text{int}}$). Using this result, the claim follows from (92) ((93)).

(iii): First, notice that $\bar{i} \in \{1, \dots, N-1\}$ and

$$h(I_{\bar{i}}) \underset{(37)}{=} h([d_{\bar{i}}, p_2)) = [h(d_{\bar{i}}), h(p_2)) \underset{(86),(87)}{=} \begin{cases} [d_{\bar{i}-1}, h(p_2)) & \text{if } \bar{i} > 1 \\ [p_1, h(p_2)) & \text{if } \bar{i} = 1\ . \end{cases} \tag{96}$$

Since $h(I_{\bar{i}}) \in \mathcal{I}_h$, it follows that

$$\underline{d} = \begin{cases} d_{\bar{i}-1} & \text{if } \bar{i} > 1 \\ p_1 & \text{if } \bar{i} = 1\ , \end{cases} \tag{97}$$

and, from (90),

$$\left.\begin{matrix} [d_{\bar{i}-1}, d_{N-1}) & \text{if } \bar{i} > 1 \\ [p_1, d_{N-1}) & \text{if } \bar{i} = 1 \end{matrix}\right\} \in \tilde{\mathcal{I}}_h = \mathcal{I}. \tag{98}$$

Since, for $\bar{i} > 1$, the only interval in $\mathcal{I}$ with lower bound $d_{\bar{i}-1}$ is $I_{\bar{i}-1}$, and the only interval in $\mathcal{I}$ with lower bound p_1, is I_N,

$$I_{\bar{i}-_N 1} = \begin{cases} I_{\bar{i}-1} & \text{if } \bar{i} > 1 \\ I_N & \text{if } \bar{i} = 1 \end{cases} = \begin{cases} [d_{\bar{i}-1}, d_{N-1}) & \text{if } \bar{i} > 1 \\ [p_1, d_{N-1}) & \text{if } \bar{i} = 1. \end{cases}$$

(iv): Notice that $\underline{i} \in \{1, \dots, N-1\}$ and

$$\begin{aligned} h(\text{int}(I_N)) \underset{(38)}{=} h((p_1, d_{\underline{i}})) &\underset{(87),(88)}{=} \begin{cases} (h(p_1), d_{\underline{i}-1}) & \text{if } \underline{i} > 1 \\ (h(p_1), \lim_{p \nearrow d_1} h(p)) & \text{if } \underline{i} = 1\ . \end{cases} \\ &= \begin{cases} (h(p_1), d_{\underline{i}-1}) & \text{if } \underline{i} > 1 \\ (h(p_1), p_2) & \text{if } \underline{i} = 1\ . \end{cases} \end{aligned} \tag{99}$$

Since $h(\text{int}(I_N)) \in \mathcal{I}_{\text{int},h}$, it follows that

$$\bar{d} = \begin{cases} d_{\underline{i}-1} & \text{if } \underline{i} > 1 \\ p_2 & \text{if } \underline{i} = 1\ , \end{cases} \tag{100}$$

and, from (91),

$$\left.\begin{matrix} (d_{N-1}, d_{\underline{i}-1}) & \text{if } \underline{i} > 1 \\ (d_{N-1}, p_2) & \text{if } \underline{i} = 1 \end{matrix}\right\} \in \tilde{\mathcal{I}}_{\text{int},h} = \mathcal{I}_{\text{int}}. \tag{101}$$

Since the only interval in $\mathcal{I}_{\text{int}}$ with lower bound d_{N-1} is $\text{int}(I_{N-1})$,

$$I_{N-1} = \begin{cases} (d_{N-1}, d_{\underline{i}-1}) & \text{if } \underline{i} > 1 \\ (d_{N-1}, p_2) & \text{if } \underline{i} = 1. \end{cases}$$

□

Proof of Proposition 4.

Proof. [5] By Lemma 2, (i) and (ii), we know that, for any $I \in \mathcal{I}$ ($I \in \mathcal{I}_{\text{int}}$), $h(I)$ is contained in an interval of $\mathcal{I}$ ($\mathcal{I}_{\text{int}}$). Since the intervals are disjoint (Proposition 3), there is exactly one interval

[5] We present the proof simultaneously for (i) and (ii). Where required, we distinguish the two cases in the text by writing the latter case in parentheses.

that contains $h(I)$. Therefore, it suffices to only consider the lower bound of an interval to identify where the interval is mapped to.

Notice that by Proposition 1, (iv), for all $[a,b) \in \mathcal{I}$ ($(a,b) \in \mathcal{I}_{\text{int}}$),

$$h([a,b)) = [h(a), \lim_{p \nearrow b} h(p)) \tag{102}$$
$$(\ h((a,b)) = (h(a), \lim_{p \nearrow b} h(p))\ ).$$

From Algorithm 1, it follows that $h(d_i) = d_{i-1}$ for all $i \in \{2,\dots,N-1\}$. Therefore (there is exactly one interval in $\mathcal{I}$ ($\mathcal{I}_{\text{int}}$) with d_{i-1} as lower bound),

$$h(I_i) = h([d_i,*)) = [d_{i-1},*) \subseteq I_{i-1} \qquad \forall i \in \{2,\dots,N-1\} \tag{103}$$
$$(\ h(\text{int}(I_i)) = h((d_i,*)) = (d_{i-1},*) \subseteq \text{int}(I_{i-1}) \qquad \forall i \in \{2,\dots,N-1\}\ ).$$

Similarly, since $h(d_1) = h(\bar{p}+\delta) = p_1$ by the definitions of d_1 and p_1, it follows that

$$h(I_1) = h([d_1,*)) = [p_1,*) \subseteq I_N \tag{104}$$
$$(\ h(\text{int}(I_1)) = h((d_1,*)) = (p_1,*) \subseteq \text{int}(I_N)\ ).$$

From (89), it follows that $h(p_1) \in [d_{N-1},*) = I_{N-1}$ ($h(p_1) \in (d_{N-1},*) = I_{N-1}$). Therefore,

$$h(I_N) = h([p_1,*)) = [h(p_1),*) \subseteq I_{N-1} \tag{105}$$
$$(\ h(\text{int}(I_N)) = h((p_1,*)) = (h(p_1),*) \subseteq \text{int}(I_{N-1})\ ).$$

□

5 Proof of Proposition 5

Proof. To show existence of the intervals $\tilde{\mathcal{I}} = \{\tilde{I}_1,\dots,\tilde{I}_N\}$, we define intervals $\tilde{I}_i$ and prove that the properties (i)–(iv) hold for these. Let $m_1 := \bar{i}+1$ (>1). We define recursively

$$\tilde{I}_{N-1} := h^{m_1}([d_{\bar{i}}, p_2]), \tag{106}$$
$$\tilde{I}_{i-_N 1} := h(\tilde{I}_i) \qquad \forall i \in \{1,\dots,N-1\}, \tag{107}$$

where '$-_N$' is as defined in Definition 4. Notice that (106) is the map of a *closed* interval.

We first show that (i)–(iii) hold for $\tilde{I}_{N-1}$. Notice that $\bar{i} \in \{1,\dots,N-1\}$. We have

$$\begin{aligned} h([d_{\bar{i}}, p_2]) &= [h(d_{\bar{i}}), h(p_2)] && \text{(by Prop. 1, (iv))} \qquad (108)\\ &= \begin{cases} [d_{\bar{i}-1}, h(p_2)] & \text{if } \bar{i} > 1 \\ [p_1, h(p_2)] & \text{if } \bar{i} = 1 \end{cases} \\ &\subseteq \begin{cases} [d_{\bar{i}-1}, d_{N-1}) & \text{if } \bar{i} > 1 \\ [p_1, d_{N-1}) & \text{if } \bar{i} = 1 \end{cases} && \text{(by Assump. 2)} \\ &= I_{\bar{i}-_N 1} && \text{(by Lemma 2, (iii)).} \qquad (109) \end{aligned}$$

From Proposition 4, it follows that, for all $i \in \{1,\dots,N\}$ and for all $m \in \{0,1,2,\dots\}$,

$$h^m(I_i) \subseteq I_{i-_N m}, \tag{110}$$
$$h^m(\text{int}(I_i)) \subseteq \text{int}(I_{i-_N m}). \tag{111}$$

With this,

$$h^{\bar{i}}([d_{\bar{i}}, p_2]) = h^{\bar{i}-1}(h([d_{\bar{i}}, p_2])) \underset{(109)}{\subseteq} h^{\bar{i}-1}(I_{\bar{i}-_N 1}) \underset{(110)}{\subseteq} I_{(\bar{i}-_N 1)-_N(\bar{i}-1)} = I_N, \tag{112}$$

and

$$\begin{aligned}
\tilde{I}_{N-1} &= h^{m_1}([d_{\bar{i}}, p_2]) = h^{\bar{i}+1}([d_{\bar{i}}, p_2]) \subseteq h(I_N) && \text{(by (112))}\\
&= h([p_1, d_{\underline{i}}]) && \text{(by (38))}\\
&= \begin{cases} [h(p_1), d_{\underline{i}-1}) & \text{if } \underline{i} > 1 \\ [h(p_1), p_2) & \text{if } \underline{i} = 1 \end{cases} && \text{(by Prop. 1, (iv))}\\
&\subseteq \begin{cases} (d_{N-1}, d_{\underline{i}-1}) & \text{if } \underline{i} > 1 \\ (d_{N-1}, p_2) & \text{if } \underline{i} = 1 \end{cases} && (d_{N-1} < h(p_1) \text{ by Prop. 2, (i)})\\
&= \mathrm{int}(I_{N-1}) && \text{(by Lemma 2, (iv))}\\
&\subseteq I_{N-1}.
\end{aligned} \tag{113}$$

Thus, (ii) holds for $\tilde{I}_{N-1}$.

Property (i) can be seen as follows: $h([d_{\bar{i}},\ h(p_2)])$ is closed (see (108)). Since $h([d_{\bar{i}}, h(p_2)]) \subseteq I_{\underline{i}-_N 1}$ (see (109)), it follows from Proposition 1, (iv), that h is continuous and strictly monotonic increasing on $h([d_{\bar{i}}, h(p_2)])$. Similarly, by (110), $h^m([d_{\bar{i}}, h(p_2)]) = h^{m-1}(h([d_{\bar{i}}, h(p_2)])) \subseteq h^{m-1}(I_{\underline{i}-_N 1}) \subseteq I_{\underline{i}-_N m}$, $m \geq 1$; thus, h is continuous and strictly monotonic increasing on $h^m([d_{\bar{i}}, h(p_2)])$. Since, for a continuous and strictly monotonic increasing function f and $a, b \in \mathbb{R}$, $f([a, b]) = [f(a), f(b)]$ (the image of a closed interval under f is a closed interval), $h^m([d_{\bar{i}}, h(p_2)])$ is closed for any $m \geq 1$ and, in particular, for $m = m_1$.

To show (iii) for $\tilde{I}_{N-1}$, let $m_2 := N - m_1$ (≥ 0) and consider

$$h^{m_2}(\tilde{I}_{N-1}) \underset{(113)}{\subseteq} h^{m_2}(I_{N-1}) \underset{(110)}{\subseteq} I_{(N-1)-_N m_2} = I_{\bar{i}} \underset{(37)}{=} [d_{\bar{i}}, p_2) \subseteq [d_{\bar{i}}, p_2], \tag{114}$$

where we used

$$(N-1) -_N m_2 = (N-1) -_N (N-1-\bar{i}) = \mathrm{mod}(N-1-N+1+\bar{i}, N) = \bar{i}. \tag{115}$$

Property (iii) then follows by

$$h^N(\tilde{I}_{N-1}) = h^{m_1}(h^{m_2}(\tilde{I}_{N-1})) \underset{(114)}{\subseteq} h^{m_1}([d_{\bar{i}}, p_2]) \underset{(106)}{=} \tilde{I}_{N-1}. \tag{116}$$

Hence, we know that (i)–(iii) hold for $i = N-1$. We next prove (i)–(iii) for $i \in \{1, \ldots, N-2, N\}$ by induction.

Induction assumption (IA): (i)–(iii) valid for some $i \in \{1, \ldots, N-1\}$. Show that this implies the validity for $i -_N 1$.

Property (ii) holds since

$$\tilde{I}_{i-_N 1} \underset{(107)}{=} h(\tilde{I}_i) \underset{\text{IA (ii)}}{\subseteq} h(\mathrm{int}(I_i)) \underset{(111)}{\subseteq} \mathrm{int}(I_{i-_N 1}) \subseteq I_{i-_N 1}. \tag{117}$$

Since $\tilde{I}_i \subseteq I_i$ (IA (ii)), h is continuous and strictly monotonic increasing on $\tilde{I}_i$. Moreover, $\tilde{I}_i$ is closed (IA (i)). Together, this implies that the image under h, $\tilde{I}_{i-_N 1} = h(\tilde{I}_i)$, is also closed; hence, (i) is true.

Property (iii) can be seen to hold by

$$h^N(\tilde{I}_{i-_N 1}) \underset{(107)}{=} h^{N+1}(\tilde{I}_i) = h(h^N(\tilde{I}_i)) \underset{\text{IA (iii)}}{\subseteq} h(\tilde{I}_i) \underset{(110)}{=} \tilde{I}_{i-_N 1}. \tag{118}$$

This completes the proof of (i)–(iii).

To prove statement (iv), take $I_i \in \mathcal{I}$ for any $i \in \{1, \dots, N\}$. Let $m_3 := i -_N \bar{i}$ (≥ 1). Then,

$$h^{m_3}(I_i) \underset{(110)}{\subseteq} I_{i-_N m_3} = I_{i-_N(i-_N\bar{i})} = I_{\bar{i}} \underset{(37)}{=} [d_{\bar{i}}, p_2) \subseteq [d_{\bar{i}}, p_2], \tag{119}$$

and, thus,

$$h^{m_1+m_3}(I_i) = h^{m_1}(h^{m_3}(I_i)) \underset{(119)}{\subseteq} h^{m_1}([d_{\bar{i}}, p_2]) \underset{(106)}{=} \tilde{I}_{N-1}. \tag{120}$$

Let $m_4 := (N -_N i) - 1$ $(\in \{0, \dots, N-1\})$. Then,

$$\begin{aligned} h^{m_1+m_3+m_4}(I_i) = h^{m_4}(h^{m_1+m_3}(I_i)) \underset{(120)}{\subseteq} h^{m_4}(\tilde{I}_{N-1}) \underset{\text{by (107)}}{=} \tilde{I}_{(N-1)-_N m_4} \\ = \tilde{I}_{(N-1)-_N((N-_N i)-1)} = \tilde{I}_i. \end{aligned} \tag{121}$$

Now, consider different cases for i:

- $i = N$. Since $m_1 + m_3 + m_4 = (\bar{i} + 1) + (N - \bar{i}) + (N - 1) = 2N$, (iv) follows directly from (121).
- $\bar{i} < i < N$. Since $m_1 + m_3 + m_4 = (\bar{i}+1) + (i-\bar{i}) + (N-i-1) = N$, (121) reads $h^N(I_i) \subseteq \tilde{I}_i$, which implies (iv) as follows:

$$h^{2N}(I_i) = h^N(h^N(I_i)) \underset{(121)}{\subseteq} h^N(\tilde{I}_i) \underset{\text{(iii)}}{\subseteq} \tilde{I}_i. \tag{122}$$

- $1 \leq i \leq \bar{i}$. Since $m_1 + m_3 + m_4 = (\bar{i}+1) + (i-\bar{i}+N) + (N-i-1) = 2N$, (iv) follows directly from (121). □

6 Proof of Proposition 6

The following Lemma is used in the proof of Proposition 6.

Lemma 3. *For all $p \in [p_1, \bar{p}+\delta)$, there exists an $m \in \mathbb{N}$ such that*

$$p, h(p), \dots, h^{m-1}(p) < \bar{p} + \delta \quad \textit{and} \quad h^m(p) \geq \bar{p} + \delta. \tag{123}$$

Furthermore, there exists an $\bar{N} \in \mathbb{N}$ (independent of p) such that $m \leq \bar{N}$, and

$$a^{2\bar{N}} < a^2 \frac{\bar{p}+\delta}{p_1}. \tag{124}$$

The lemma says that if $p(0)$ starts anywhere in $[p_1, \bar{p}+\delta)$, there is a maximum number $\bar{N}$ of iterations (22), for which $p(k)$ remains in $[p_1, \bar{p}+\delta)$. The slope of $\bar{N}$ successive maps in $[p_1, \bar{p}+\delta)$ is bounded by (124).

Proof. Let $m \in \mathbb{N}$ such that $p, h(p), \ldots, h^{m-1}(p) < \bar{p}+\delta$ (such an m exists since $p < \bar{p}+\delta$). Then, from (23), for all $1 \leq \ell \leq m$,

$$h^{\ell}(p) = a^2\, h^{\ell-1}(p) + 1 > a^2\, h^{\ell-1}(p), \tag{125}$$

and, therefore,

$$h^{\ell}(p) > a^{2\ell}\, p. \tag{126}$$

Since, $|a| > 1$, $\lim_{m\to\infty} a^{2m} p = \infty$. Hence, there exists an m such that $h^m(p) \geq \bar{p}+\delta$ and (123) holds. Note that m depends on p.

Now, we seek the largest possible integer m such that (123) holds. Since $h^{\ell}(p_1) \leq h^{\ell}(p)$ for all $p \in [p_1, \bar{p}+\delta)$ and $\ell \leq m$, the greatest m such that (123) holds is $\bar{N} \in \mathbb{N}$ defined by

$$p_1, h(p_1), \ldots, h^{\bar{N}-1}(p_1) < \bar{p}+\delta \quad \text{and} \quad h^{\bar{N}}(p_1) \geq \bar{p}+\delta. \tag{127}$$

Hence, $\bar{N}$ is independent of p, and $m \leq \bar{N}$. From (126) and (127), it follows that

$$a^{2(\bar{N}-1)}\, p_1 < h^{\bar{N}-1}(p_1) < \bar{p}+\delta \quad \underset{(p_1>0,\, a^2>0)}{\Rightarrow} \quad a^{2\bar{N}} < a^2 \frac{\bar{p}+\delta}{p_1}.$$

□

Proof of Proposition 6.

Proof. Take any $I_i \in \mathcal{I}$ and any $\tilde{p} \in \mathrm{int}(I_i)$.

Differentiability: By Proposition 1, (v), h is differentiable for any $p \in \mathrm{int}(I)$, $I \in \mathcal{I}$. So, in particular, h is differentiable at $\tilde{p}$. We prove by induction that h^j is differentiable at $\tilde{p}$ for all $j \geq 1$.

Induction assumption (IA): h^j is differentiable at $\tilde{p}$. By the chain rule, [3], $h^{j+1}(\tilde{p}) = h(h^j(\tilde{p}))$ is differentiable at $\tilde{p}$ if h^j is differentiable at $\tilde{p}$ (IA) and h is differentiable at $h^j(\tilde{p})$. From Proposition 4, (ii), (or equation (111)) it follows that

$$h^j(\tilde{p}) \in \mathrm{int}(I_{i-_N j}). \tag{128}$$

Since h is differentiable on any $\mathrm{int}(I)$ with $I \in \mathcal{I}$ (so, in particular, on $\mathrm{int}(I_{i-_N j})$), the differentiability of h^{j+1} at $\tilde{p}$ follows.

Contraction mapping: By the chain rule,

$$\begin{aligned}(h^N)'(\tilde{p}) &= h'(h^{N-1}(\tilde{p})) \cdot (h^{N-1})'(\tilde{p}) \\ &= h'(h^{N-1}(\tilde{p})) \cdot h'(h^{N-2}(\tilde{p})) \cdot (h^{N-2})'(\tilde{p}) \\ &= h'(h^{N-1}(\tilde{p})) \cdot h'(h^{N-2}(\tilde{p})) \cdot \ldots \cdot h'(h(\tilde{p})) \cdot h'(\tilde{p}) \\ &= \prod_{j=0}^{N-1} h'(h^j(\tilde{p})) = \prod_{p\in\mathcal{P}} h'(p),\end{aligned} \tag{129}$$

with

$$\mathcal{P} := \{\tilde{p}, h(\tilde{p}), \ldots, h^{N-1}(\tilde{p})\}. \tag{130}$$

Notice from (128) for $j = 0, 1, \ldots, N-1$ that, for every point $p \in \mathcal{P}$, there is exactly one interval $I \in \mathcal{I}$ such that $p \in \mathrm{int}(I)$.

Let $\mathcal{I}_L \subset \mathcal{I}$ denote the set of all intervals $I \in \mathcal{I}$ with $I < \bar{p}+\delta$ (intervals left of the discontinuity $\bar{p}+\delta$), and let $\mathcal{I}_R \subset \mathcal{I}$ denote the set of all $I \in \mathcal{I}$ with $I \geq \bar{p}+\delta$ (intervals right of the discontinuity $\bar{p}+\delta$). Furthermore, let N_L and N_R denote the number of elements in $\mathcal{I}_L$ and $\mathcal{I}_R$, respectively. Notice that $N_L \geq 1$ and $N_R \geq 1$ by the construction of the intervals. Then,

$$h'(p) = a^2 \qquad \forall p \in \mathrm{int}(I),\, I \in \mathcal{I}_L, \tag{131}$$

which follows directly from (23); and

$$h'(p) = g'(p) < g'(\bar{p}+\delta) \qquad \forall p \in \mathrm{int}(I),\, I \in \mathcal{I}_R, \tag{132}$$

where the inequality follows from g' being strictly monotonically decreasing, which is seen from

$$g''(p) = -\frac{2a^2c^2}{(c^2p+1)^3} < 0. \tag{133}$$

With these results, it follows from (129) that

$$(h^N)'(\tilde{p}) < a^{2N_L}\left(g'(\bar{p}+\delta)\right)^{N_R}. \tag{134}$$

Since $a^2 > 1$ and $g'(\bar{p}+\delta) < 1$, it depends on the ratio of N_R to N_L whether the map h^N is contractive. We investigate this ratio next.

Define a subset $\underline{\mathcal{I}} \subset \mathcal{I}$ as a maximum successive sequence of M intervals all being left of $\bar{p}+\delta$:

$$\begin{aligned} \underline{\mathcal{I}} :=& \big(I_\ell, I_{\ell -_N 1}, \ldots, I_{\ell -_N (M-1)}\big) \\ & \text{such that } I_\ell, I_{\ell -_N 1}, \ldots, I_{\ell -_N (M-1)} \in \mathcal{I}_L, \quad M \leq N_L, \\ & \text{and } I_{\ell +_N 1}, I_{\ell -_N M} \in \mathcal{I}_R, \end{aligned} \tag{135}$$

where '$+_N$' is analogously defined to '$-_N$' in Definition 4:

$$\alpha +_N \beta = \begin{cases} \mathrm{mod}(\alpha+\beta, N) & \text{if } \mathrm{mod}(\alpha+\beta, N) > 0 \\ N & \text{if } \mathrm{mod}(\alpha+\beta, N) = 0, \end{cases} \tag{136}$$

for $\alpha, \beta \in \mathbb{Z}$ and $N \in \mathbb{N}$. Let there be $\kappa \geq 1$ distinct interval subsequences (135), which we call $\underline{\mathcal{I}}_1, \ldots, \underline{\mathcal{I}}_\kappa$ with $M_1, \ldots, M_\kappa$ their numbers of elements, respectively. An example with two interval subsequences $\underline{\mathcal{I}}_1$, $\underline{\mathcal{I}}_2$ is provided in Fig. 11. Notice that $N_L = M_1 + \cdots + M_\kappa$.

Using Lemma 3, it can be shown by contradiction that $M_j \leq \bar{N}$ for all $j \leq \kappa$, where $\bar{N}$ is as defined in Lemma 3. Assume $M_j > \bar{N}$. Then, there exists $I_\ell \in \mathcal{I}$ and $p \in I_\ell$ such that $p, h(p), \ldots, h^{M_j-1}(p) < \bar{p}+\delta$ and $h^{M_j} \geq \bar{p}+\delta$. But, from Lemma 3, it then follows that $M_j \leq \bar{N}$, which contradicts the assumption.

From $M_j \leq \bar{N}, j \leq \kappa$, it follows that

$$N_L = M_1 + \cdots + M_\kappa \leq \kappa\bar{N}. \tag{137}$$

For each subsequence of intervals $\underline{\mathcal{I}}_j$, $j \leq \kappa$, there is at least one distinct interval $I \in \mathcal{I}_R$ (namely, $I_{\ell -_N M}$); hence,

$$N_R \geq \kappa. \tag{138}$$

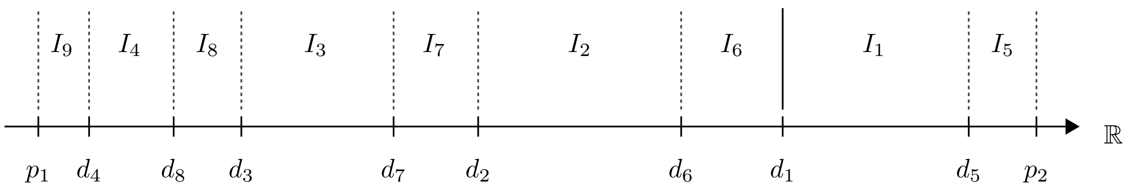

Figure 11: Illustration of the intervals $\mathcal{I}$ obtained for the parameter values $a = 1.2$, $c = 1$, and $\delta = 9.6$ (for better visibility the relative scaling of the intervals has been adapted). There are two distinct interval subsequences satisfying (135): $\underline{\mathcal{I}}_1 = (I_4, I_3, I_2)$ and $\underline{\mathcal{I}}_2 = (I_9, I_8, I_7, I_6)$.

Combining (137) and (138), we obtain a bound on the ratio of N_L and N_R,

$$N_L \leq \kappa \bar{N} \leq N_R \bar{N}. \tag{139}$$

With this result, we can rewrite (134),

$$\begin{aligned}(h^N)'(\tilde{p}) &< a^{2N_L} \left(g'(\bar{p}+\delta)\right)^{N_R} \\ &\leq a^{2N_L}\, a^{2(N_R\bar{N}-N_L)} \left(g'(\bar{p}+\delta)\right)^{N_R} \\ &= \left(a^{2\bar{N}}\, g'(\bar{p}+\delta)\right)^{N_R}.\end{aligned} \tag{140}$$

We show below that $a^{2\bar{N}}\, g'(\bar{p}+\delta) < 1$. With this, the statement of Proposition 6 follows from (140) with $L := (a^{2\bar{N}}\, g'(\bar{p}+\delta))^{N_R} < 1$.

It thus remains to show that

$$a^{2\bar{N}}\, g'(\bar{p}+\delta) < 1. \tag{141}$$

First, notice from Lemma 3 that

$$a^{2\bar{N}}\, g'(\bar{p}+\delta) < a^{2\frac{\bar{p}+\delta}{p_1}}\, g'(\bar{p}+\delta) \underset{(51)}{=} \frac{a^4\,(\bar{p}+\delta)}{p_1\,(c^2(\bar{p}+\delta)+1)^2}. \tag{142}$$

Recall that

$$p_1 = h(\bar{p}+\delta) = g(\bar{p}+\delta) \underset{(50)}{=} \frac{(a^2+c^2)(\bar{p}+\delta)+1}{c^2(\bar{p}+\delta)+1}, \tag{143}$$

and that $\bar{p}$ is the positive solution of (6) (with $q = r = 1$), which is given explicitly by

$$\bar{p} = \frac{a^2 - 1 + c^2 + S}{2c^2} > 0 \tag{144}$$

with $S := \sqrt{(a^2-1+c^2)^2 + 4c^2} > 0$. With (143) and (144), the right-hand side of (142) can be rewritten,

$$\begin{aligned}\frac{a^4\,(\bar{p}+\delta)}{p_1\,(c^2(\bar{p}+\delta)+1)^2} &= \frac{a^4\,(\bar{p}+\delta)}{\left((a^2+c^2)(\bar{p}+\delta)+1\right)\left(c^2(\bar{p}+\delta)+1\right)} \\ &= \frac{a^4\left(\frac{a^2-1+c^2+S}{2c^2}+\delta\right)}{\left((a^2+c^2)\left(\frac{a^2-1+c^2+S}{2c^2}+\delta\right)+1\right)\left(c^2\left(\frac{a^2-1+c^2+S}{2c^2}+\delta\right)+1\right)} \\ &= \frac{\text{NUM}}{\text{DEN}}\end{aligned} \tag{145}$$

with[6]

$$\begin{aligned}\text{NUM} &:= 4c^2 \cdot a^4 \Big(\tfrac{(a^2-1)+c^2+S}{2c^2} + \delta\Big) \qquad (146)\\ &= 2Sa^4 - 2a^4 + 2a^6 + 2a^4c^2 + 4a^4c^2\delta \\ \text{DEN} &:= 4c^2 \cdot \Big((a^2+c^2)\Big(\tfrac{a^2-1+c^2+S}{2c^2} + \delta\Big) + 1\Big)\Big(c^2\Big(\tfrac{a^2-1+c^2+S}{2c^2} + \delta\Big) + 1\Big) \qquad (147)\\ &= S^2a^2 + S^2c^2 + 2Sa^4 + 4Sa^2c^2\delta + 4Sa^2c^2 + 4Sc^4\delta + 2Sc^4 \\ &\quad + 2Sc^2 + a^6 + 4a^4c^2\delta + 3a^4c^2 + 4a^2c^4\delta^2 + 8a^2c^4\delta + 3a^2c^4 \\ &\quad + 2a^2c^2 - a^2 + 4c^6\delta^2 + 4c^6\delta + c^6 + 4c^4\delta + 2c^4 + c^2.\end{aligned}$$

Since DEN > 0 (can be seen from (147) and $a^2 > 0$, $c^2 > 0$, $\delta > 0$, $S > 0$, and $a^2 - 1 > 0$),

$$\frac{\text{NUM}}{\text{DEN}} < 1 \quad \Leftrightarrow \quad \text{DEN} - \text{NUM} > 0. \tag{148}$$

Using $S^2 = (a^2 - 1 + c^2)^2 + 4c^2$, we get[6]

$$\begin{aligned}\text{DEN} - \text{NUM} = {} & 2Sc^2 + 2Sc^4 + 4c^4\delta + 4c^6\delta + 2c^2 + 4c^4 + 2c^6 + 2a^2c^2 \\ & + 6a^2c^4 + 4a^4c^2 + 4c^6\delta^2 + 4Sa^2c^2 + 8a^2c^4\delta + 4a^2c^4\delta^2 \\ & + 4Sc^4\delta + 4Sa^2c^2\delta. \end{aligned}\tag{149}$$

Since $a^2 > 0$, $c^2 > 0$, $\delta > 0$, and $S > 0$, all summands in (149) are positive. Hence, DEN$-$NUM > 0, and (141) follows from (142), (145), and (148). □

7 Proof of Corollary 2

Proof. Take any $\tilde{I}_i \in \tilde{\mathcal{I}}$ and any $p, \tilde{p} \in \tilde{I}_i$. Without loss of generality, $\tilde{p} < p$ (for $p = \tilde{p}$ the statement holds trivially). By Proposition 2, (ii), and 5, (ii), h^N is continuous on $[\tilde{p}, p]$ and, by Proposition 6, h^N is differentiable on $(\tilde{p}, p)$. The mean value theorem, [3], assures the existence of a $\xi \in (p, \tilde{p})$ such that

$$\frac{h^N(p) - h^N(\tilde{p})}{p - \tilde{p}} = (h^N)'(\xi). \tag{150}$$

Therefore, with Proposition 6,

$$|h^N(p) - h^N(\tilde{p})| = \big|(h^N)'(\xi)\big|\,|p - \tilde{p}| \leq L|p - \tilde{p}|.$$ □

[6] A MATLAB program performing the algebraic manipulations is available at `www.cube.ethz.ch/downloads`.

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