Appendix: Derivation of the Equations of Motion

Recall that the Lagrangian is given by $\mathcal{L}: \; T(SO(3) \times \mathbb{T}^3) \rightarrow \mathbb{R}$,

$$\mathcal{L}(R_{IK}, \psi, \dot{R}_{IK}, \dot{\psi}) = \frac{1}{2}\omega_h{}^{\mathrm{T}}\hat{\Theta}_0\omega_h + \frac{1}{2}\;(\omega_h + \omega_w)^{\mathrm{T}} \Theta_w(\omega_h + \omega_w) + m^{\mathrm{T}}g, \tag{29}$$

with $\tilde{\omega}_h := R_{IK}^{\mathrm{T}}\dot{R}_{IK}$ and $\omega_w := \dot{\psi}$, and where the tilde denotes the skew symmetric operator mapping from $\mathbb{R}^3$ to so(3) such that $\tilde{v}a = v \times a$, $\forall a \in \mathbb{R}^3$. The configuration of the Cubli is therefore given by the attitude, expressed by the rotation matrix $R_{IK} \in SO(3)$, and the wheel positions parametrized by $\psi \in \mathbb{T}^3$. For the definitions of the notations please refer to the corresponding paper.

The principle of virtual action, i.e. the virtual work integrated over time, states that the solutions to the equation of motion (expressed in local coordinates by q, $\dot{q}$) have to fulfill

$$\int_{t_0}^{t_e} \left(\delta\mathcal{L}(q, \dot{q}) + \delta q^{\mathrm{T}} f_{NP}\right)\; dt = 0, \tag{30}$$

for all variations δq which vanish at the endpoints, i.e. at $t = t_0$ and $t = t_e$. Since $R_{IK} \in SO(3)$, variations of R_{IK} are constrained by

$$\delta R_{IK}^{\mathrm{T}} R_{IK} + R_{IK}^{\mathrm{T}}\delta R_{IK} = 0$$

and implies that $R_{IK}^{\mathrm{T}}\delta R_{IK}$ is skew symmetric. Therefore δR_{IK} is parametrized by $\delta R_{IK} = R_{IK}\delta\tilde{\Sigma}$, with $\Sigma \in \mathbb{R}^3$. This allows to express the variations $\delta\omega_h$ by

$$\begin{aligned}\delta\tilde{\omega}_h &= \delta R_{IK}^{\mathrm{T}}\dot{R}_{IK} + R_{IK}^{\mathrm{T}}\delta\dot{R}_{IK} \\ &= -\delta\tilde{\Sigma}\tilde{\omega}_h + \tilde{\omega}_h\delta\tilde{\Sigma} + \delta\dot{\tilde{\Sigma}}\end{aligned}$$

and since it holds for the commutator $[A, B] = AB - BA$ that $[\tilde{a}, \tilde{b}] = (a \times b)\tilde{}$ we obtain

$$\delta\omega_h = \delta\dot{\Sigma} + \omega_h \times \delta\Sigma. \tag{31}$$

The virtual work of the non-potential forces, that is the torque T applied to the reaction wheels, is given by $\delta W_z = \delta\psi^{\mathrm{T}}T$.

Hence, from the principle of virtual action (30) with the Lagrangian given by (29) it follows that

$$\int_{t_0}^{t_e} \left(\frac{\partial\mathcal{L}}{\partial\omega_h}\delta\omega_h + \frac{\partial\mathcal{L}}{\partial\omega_w}\delta\omega_w + m^{\mathrm{T}}\delta\tilde{\Sigma}^{\mathrm{T}}g + \delta\psi^{\mathrm{T}}T\right) dt = 0,$$

for all variations $\delta\Sigma \in \mathbb{R}^3$ and $\delta\psi \in \mathbb{R}^3$. Note that the tangent space of $SO(3)$ and $\mathbb{T}^3$ are both isomorphic to $\mathbb{R}^3$. Inserting the expression of the variation of $\delta\omega_h$ and using integration by parts allows to state that

$$\int_{t_0}^{t_e} \left[\left(-\dot{p}_{\omega_h}^{\mathrm{T}} + p_{\omega_h}^{\mathrm{T}}\tilde{\omega}_h - g^{\mathrm{T}}\tilde{m}\right)\delta\Sigma + \left(T^{\mathrm{T}} - \dot{p}_{\omega_w}^{\mathrm{T}}\right)\delta\psi\right] dt$$

has to vanish for all $\delta\Sigma \in \mathbb{R}^3$ and for all $\delta\psi \in \mathbb{R}^3$. Note that the partial derivatives of the Lagrangian have been replaced by the generalized momenta,

$$p_{\omega_h} := \frac{\partial\mathcal{L}}{\partial\omega_h}^{\mathrm{T}} = \Theta_0\omega_h + \Theta_w\omega_w, \tag{32}$$

$$p_{\omega_w} := \frac{\partial\mathcal{L}}{\partial\omega_w}^{\mathrm{T}} = \Theta_w(\omega_h + \omega_w). \tag{33}$$

From the fundamental lemma of the calculus of variations it follows that

$$\dot{p}_{\omega_h} = -\tilde{\omega}_h p_{\omega_h} + \tilde{m}g, \tag{34}$$

$$\dot{p}_{\omega_w} = T. \tag{35}$$

Since it holds for the gravity vector $\vec{g}$ that ${}^B(\dot{\vec{g}}) = 0 = \dot{g} + \omega_h \times g$, the motion of the Cubli can be described by

$$\dot{g} = -\tilde{\omega}_h g, \tag{36}$$

$$\dot{p}_{\omega_h} = -\tilde{\omega}_h p_{\omega_h} + \tilde{m}g, \tag{37}$$

$$\dot{p}_{\omega_w} = T. \tag{38}$$

However, this represents a reduced description of the attitude, as the orientation around the gravity vector is not captured.