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# A Limiting Property of the Matrix Exponential with Application to Multi-loop Control

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***Abstract*— A limiting property of the matrix exponential is proven: For a real square matrix, where the log norm of the upper-left n by n block approaches negative infinity in a limiting process, the matrix exponential goes to zero in the first n rows and n columns. This property is useful for simplification of dynamic systems that exhibit modes with sufficiently different time scales; for example, in multi-loop control systems with fast inner and slow outer feedback loops. For this case, we derive a time scale separation algorithm for a linear continuous-time model under the assumption of high-gain inner loop feedback, which yields a simplified discrete-time model at the slow time scale. The proposed technique is applied to the design of a two-loop control system for stabilizing an inverted pendulum. Experimental results are provided.**

***Index Terms*— matrix exponential, limiting property, logarithmic norm, time scale separation, multi-loop control**

## I. Introduction

The matrix exponential arises naturally when solving the linear ordinary differential equation (ODE)

$$\dot{x}(t) = Ax(t), \quad x(0) = x_0; \tag{1}$$

the solution is

$$x(t) = \exp(At)x_0. \tag{2}$$

In particular, when discretizing the system (1) at a sampling rate $T$, the state transition matrix is given by the matrix exponential,

$$x(t+T) = \exp(AT)x(t). \tag{3}$$

Because of its relevance in various fields, the matrix exponential and its properties have been subject to extensive studies, for example [1]–[4].

In many dynamic systems, there are parts that operate at different time scales. Therefore, it is interesting to ask whether the matrix exponential in (3) can be simplified if the system (1) exhibits modes with sufficiently large time scale separation. In particular, what happens in the limiting case of some infinitely fast modes, i.e. the case of infinite time scale separation? We prove in this paper that, if some of the eigenvalues of a general real matrix approach negative infinity in a particular manner, the corresponding rows and columns of the matrix exponential go to 0. Applying this to a system like (1), one achieves a significantly simpler system representation.

This consideration is of particular practical relevance for a cascaded control architecture with fast inner and slow outer loops. Especially, one may not have full knowledge of the inner loop (for example, when using off-the-shelf control components); however, one needs to approximatively incorporate the inner loop behavior in a model used for the design of an outer loop controller. Based on the limiting property of the matrix exponential, we present a time scale separation technique that results in a model of the multi-loop system at the slow time scale relevant to outer loop controller. It incorporates the behavior of the inner loop by assuming that the inner loop is an ideal feedback loop, i.e. it is arbitrarily fast.

We apply the proposed technique to the design of a two-loop control system for balancing an inverted pendulum. As actuator, we use a DC motor with a built-in local velocity controller. Our objective is to design an outer loop controller operating at a slower rate that stabilizes the system. We provide experimental results of the closed-loop operation.

The paper is organized as follows: In Sec. II, we state and prove the limiting property of the matrix exponential. Using this result, we derive the time scale separation algorithm for a multi-loop system with high-gain inner feedback loops in Sec. III. In Sec. IV, the proposed technique is applied to the controller design for the inverted pendulum. We conclude with some remarks in Sec. V.

## II. Main Result

We will work exclusively with the vector 2-norm and its induced matrix norm, i.e., for a vector $x \in \mathbb{R}^n$ and a real matrix $M$,

$$\|x\| = \Big(\sum_{i=1}^{n} |x_i|^2\Big)^{1/2}, \quad \|M\| = \max_{\|x\|=1} \|Mx\|.$$

Let $\mu(M)$ denote the *log norm* of $M$ (associated with the 2-norm), [1], [2],

$$\mu(M) := \max\{\mu | \mu \text{ an eigenvalue of } (M+M^T)/2\},$$

where the matrix $M^T$ denotes the transpose of $M$. We shall exploit the following properties of the log norm $\mu(M)$, [2],

$$\|e^{Mt}\| \leq e^{\mu(M)t} \tag{4}$$

$$\mu(M) \leq \|M\| \tag{5}$$

$$\mu(M+P) \leq \mu(M) + \|P\|, \tag{6}$$

where $M$, $P$ are real square matrices and $t \geq 0$.

### A. Theorem

The following theorem states the main result of this paper:

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*Theorem 2.1:* Let $M$ be a real square matrix and $K_i$, $i = 1, 2, \ldots, \infty$ be a sequence of real square matrices. If $\lim_{i\to\infty} \mu(-K_i) = -\infty$, then

$$\lim_{i\to\infty} \exp\left(\begin{bmatrix} M_{11} - K_i & M_{12} \\ M_{21} & M_{22} \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & \exp(M_{22}) \end{bmatrix}.$$

Note that it would be relatively straightforward to prove this result if the matrices

$$\begin{bmatrix} M_{11} & 0 \\ M_{21} & M_{22} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -K_i & M_{12} \\ 0 & 0 \end{bmatrix} \tag{7}$$

commuted. Then

$$\begin{aligned}
&\lim_{i\to\infty} \exp\left(\begin{bmatrix} M_{11} - K_i & M_{12} \\ M_{21} & M_{22} \end{bmatrix}\right) \\
&= \lim_{i\to\infty} \exp\left(\begin{bmatrix} M_{11} & 0 \\ M_{21} & M_{22} \end{bmatrix}\right) \exp\left(\begin{bmatrix} -K_i & M_{12} \\ 0 & 0 \end{bmatrix}\right) \\
&= \begin{bmatrix} \exp(M_{11}) & 0 \\ \bar{M}_{21} & \exp(M_{22}) \end{bmatrix} \lim_{i\to\infty} \begin{bmatrix} \exp(-K_i) & \bar{M}_{12,i} \\ 0 & I \end{bmatrix} \\
&= \begin{bmatrix} \exp(M_{11}) & 0 \\ \bar{M}_{21} & \exp(M_{22}) \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 \\ 0 & \exp(M_{22}) \end{bmatrix},
\end{aligned}$$

where, [4],

$$\begin{aligned}
\bar{M}_{21} &:= \int_0^1 e^{M_{11}(1-\tau)} M_{21} e^{M_{22}\tau}\, d\tau, \\
\bar{M}_{12,i} &:= \int_0^1 e^{-K_i(1-\tau)} M_{12}\, d\tau,
\end{aligned}$$

and we used $\lim_{i\to\infty} \exp(-K_i) = 0$ and $\lim_{i\to\infty} \bar{M}_{12,i} = 0$, since

$$\begin{aligned}
\|e^{-K_i}\| &\le e^{\mu(-K_i)} \to 0 \text{ as } i \to \infty, \\
\|\bar{M}_{12,i}\| &\le \int_0^1 e^{\mu(-K_i)(1-\tau)} \|M_{12}\|\, d\tau \\
&= \frac{\|M_{12}\|}{\mu(-K_i)} \left(e^{\mu(-K_i)} - 1\right) \to 0 \text{ as } i \to \infty.
\end{aligned}$$

In general, however, the matrices (7) do not commute; the proof of the theorem for the general case is a little more involved.

### B. Proof

We need the following Gronwall-type inequality, [5], [6]:

*Lemma 2.1:* Let $u(t)$, $\alpha(t)$ be continuous functions in $J = [t_0, t_1]$, let $\beta(t)$ be a nonnegative continuous function in $J$, let $\kappa(t, s)$ be a nonnegative continuous function for $t_0 \le s \le t \le t_1$, and suppose

$$u(t) \le \alpha(t) + \beta(t) \int_{t_0}^t \kappa(t, s) u(s)\, ds, \quad t \in J.$$

Then

$$u(t) \le \bar{\alpha}(t) \exp\left(\bar{\beta}(t) \int_{t_0}^t \bar{\kappa}(t, s)\, ds\right), \quad t \in J,$$

where

$$\begin{aligned}
\bar{\alpha}(t) &:= \sup_{\tau\in[t_0,t]} \alpha(\tau), \quad \bar{\beta}(t) := \sup_{\tau\in[t_0,t]} \beta(\tau), \\
\bar{\kappa}(t, s) &:= \sup_{\tau\in[s,t]} \kappa(\tau, s).
\end{aligned}$$

As an intermediate step of the proof, we will use the following matrix ODE in $X(\cdot)$ and $Y(\cdot)$,

$$\begin{aligned}
\dot{X}(t) &= (M_{11} - K_i) X(t) + M_{12} Y(t), \\
\dot{Y}(t) &= M_{21} X(t) + M_{22} Y(t), \\
X(0) &= X_0, \quad Y(0) = Y_0,
\end{aligned} \tag{8}$$

with real matrices $M_{11}$, $M_{12}$, $M_{21}$, $M_{22}$, $K_i$ and initial conditions $X_0$ and $Y_0$.

The proof of Theorem 2.1 is organized into three parts:

1) For the system (8), with initial conditions $X_0 = I$ and $Y_0 = 0$, show that, if $\lim_{i\to\infty} \mu(-K_i) = -\infty$, then, for all (finite) $t > 0$,

$$\lim_{i\to\infty} X(t) = 0 \quad \text{and} \quad \lim_{i\to\infty} Y(t) = 0. \tag{9}$$

2) For the system (8), with initial conditions $X_0 = 0$ and $Y_0 = I$, show that, if $\lim_{i\to\infty} \mu(-K_i) = -\infty$, then, for all (finite) $t > 0$,

$$\lim_{i\to\infty} X(t) = 0 \quad \text{and} \quad \lim_{i\to\infty} Y(t) = \exp(M_{22} t). \tag{10}$$

3) Consider the matrix ODE that is solved uniquely by the matrix exponential

$$\exp\left(\begin{bmatrix} M_{11} - K_i & M_{12} \\ M_{21} & M_{22} \end{bmatrix} t\right)$$

and reformulate it into the form (8); then use the results of Part 1 and Part 2 to conclude Theorem 2.1.

*Proof:* Since $\lim_{i\to\infty} \mu(-K_i) = -\infty$, there exists $i_0 \in \mathbb{N}$ such that for all $i \ge i_0$

$$\mu(M_{11} - K_i) \le \|M_{11}\| + \mu(-K_i) < 0, \tag{11}$$

$$\mu(M_{11} - K_i) - \|M_{22}\| < -1. \tag{12}$$

In the following, we consider sufficiently large $i$ such that $i \ge i_0$.

PART 1: The unique solution to (8) is, for all $t \ge 0$,

$$X(t) = e^{(M_{11}-K_i)t} X_0 + \int_0^t e^{(M_{11}-K_i)(t-\tau)} M_{12} Y(\tau)\, d\tau \tag{13}$$

$$Y(t) = e^{M_{22}t} Y_0 + \int_0^t e^{M_{22}(t-\tau)} M_{21} X(\tau)\, d\tau. \tag{14}$$

Substituting (13) into (14) and using the initial conditions $X_0 = I$ and $Y_0 = 0$ yields

$$\begin{aligned}
Y(t) &= \int_0^t e^{M_{22}(t-\tau)} M_{21} e^{(M_{11}-K_i)\tau}\, d\tau \\
&+ \int_0^t \int_0^\tau e^{M_{22}(t-\tau)} M_{21} e^{(M_{11}-K_i)(\tau-s)} M_{12} Y(s)\, ds\, d\tau \\
&= \int_0^t e^{M_{22}(t-\tau)} M_{21} e^{(M_{11}-K_i)\tau}\, d\tau \\
&+ \int_0^t \int_s^t e^{M_{22}(t-\tau)} M_{21} e^{(M_{11}-K_i)(\tau-s)} M_{12} Y(s)\, d\tau\, ds,
\end{aligned} \tag{15}$$

where the order of integration in the last term was interchanged. This is valid (see e.g. [7]), since the integrand is continuous and we can express the integration region in either of the two ways: $\{(\tau,s): 0 \le \tau \le t, 0 \le s \le \tau\}$ or $\{(\tau,s): 0 \le s \le t, s \le \tau \le t\}$.

Using (4), (5), we obtain the inequality

$$\begin{aligned}
\|Y(t)\| &\le \|M_{21}\| \int_0^t \|e^{M_{22}(t-\tau)}\| \|e^{(M_{11}-K_i)\tau}\| \, d\tau \\
&+ \int_0^t \|M_{21}\|\|M_{12}\| \int_s^t \|e^{M_{22}(t-\tau)}\|\|e^{(M_{11}-K_i)(\tau-s)}\| \, d\tau \, \|Y(s)\| \, ds \\
&\le \|M_{21}\| \int_0^t e^{\|M_{22}\|(t-\tau)} e^{\mu(M_{11}-K_i)\tau} \, d\tau \\
&+ \int_0^t \|M_{21}\|\|M_{12}\| \int_s^t e^{\|M_{22}\|(t-\tau)} e^{\mu(M_{11}-K_i)(\tau-s)} \, d\tau \, \|Y(s)\| \, ds
\end{aligned} \tag{16}$$

$$= \alpha(t) + \int_0^t \kappa(t,s)\|Y(s)\| \, ds, \tag{17}$$

where

$$\alpha(t) := \|M_{21}\| \int_0^t e^{\|M_{22}\|(t-\tau)} e^{\mu(M_{11}-K_i)\tau} \, d\tau \tag{18}$$

$$\kappa(t,s) := \|M_{21}\|\|M_{12}\| \int_s^t e^{\|M_{22}\|(t-\tau)} e^{\mu(M_{11}-K_i)(\tau-s)} \, d\tau. \tag{19}$$

Applying Lemma 2.1 to (17) yields, for all finite $t \ge 0$,

$$\|Y(t)\| \le \bar{\alpha}(t) \exp\left( \int_0^t \bar{\kappa}(t,s) \, ds \right), \tag{20}$$

where

$$\bar{\alpha}(t) = \sup_{\tau\in[0,t]} \alpha(\tau), \quad \bar{\kappa}(t,s) = \sup_{\tau\in[s,t]} \kappa(\tau,s).$$

Next, we derive bounds for $\alpha(t)$, $\bar{\alpha}(t)$ and $\kappa(t,s)$, $\bar{\kappa}(t,s)$ using the properties (11), (12). First, consider (18),

$$\begin{aligned}
\alpha(t) &= \|M_{21}\| e^{\|M_{22}\|t} \int_0^t e^{(\mu(M_{11}-K_i)-\|M_{22}\|)\tau} \, d\tau \\
&= \frac{\|M_{21}\|}{\mu_i} \big( e^{\|M_{22}\|t} - \underbrace{e^{\mu(M_{11}-K_i)t}}_{\in(0,1]} \big) \\
&\le \frac{\|M_{21}\|}{\mu_i} e^{\|M_{22}\|t} = \frac{M_1(t)}{\mu_i},
\end{aligned}$$

where $\mu_i := \|M_{22}\| - \mu(M_{11}-K_i) > 1$ and $M_1(t) := \|M_{21}\| e^{\|M_{22}\|t} \ge 0$ is a continuous function in $t$. Therefore,

$$\begin{aligned}
\bar{\alpha}(t) &= \sup_{\tau\in[0,t]} \alpha(\tau) \le \sup_{\tau\in[0,t]} \frac{\|M_{21}\|}{\mu_i} e^{\|M_{22}\|\tau} \\
&= \frac{\|M_{21}\|}{\mu_i} e^{\|M_{22}\|t} = \frac{M_1(t)}{\mu_i}.
\end{aligned} \tag{21}$$

Similarly, we obtain for (19), with $s \le t$,

$$\begin{aligned}
\kappa(t,s) &= \|M_{21}\|\|M_{12}\| e^{\|M_{22}\|t} e^{-\mu(M_{11}-K_i)s} \\
&\quad \int_s^t e^{(\mu(M_{11}-K_i)-\|M_{22}\|)\tau} \, d\tau \\
&= \frac{\|M_{21}\|\|M_{12}\|}{\mu_i} \big( e^{\|M_{22}\|(t-s)} - \underbrace{e^{\mu(M_{11}-K_i)(t-s)}}_{\in(0,1]} \big) \\
&\le \frac{\|M_{21}\|\|M_{12}\|}{\mu_i} e^{\|M_{22}\|t} = \frac{M_2(t)}{\mu_i},
\end{aligned}$$

where $M_2(t) := \|M_{21}\|\|M_{12}\| e^{\|M_{22}\|t} \ge 0$ is a continuous function in $t$. Therefore,

$$\begin{aligned}
\bar{\kappa}(t,s) &= \sup_{\tau\in[s,t]} \kappa(\tau,s) \le \sup_{\tau\in[s,t]} \frac{\|M_{21}\|\|M_{12}\|}{\mu_i} e^{\|M_{22}\|\tau} \\
&= \frac{\|M_{21}\|\|M_{12}\|}{\mu_i} e^{\|M_{22}\|t} = \frac{M_2(t)}{\mu_i}.
\end{aligned} \tag{22}$$

With (21) and (22), we can now bound (20),

$$\begin{aligned}
\|Y(t)\| &\le \frac{M_1(t)}{\mu_i} \exp\left( \int_0^t \frac{M_2(t)}{\mu_i} \, ds \right) \\
&\le \frac{M_1(t)}{\mu_i} \exp\left( \frac{M_2(t)}{\mu_i} t \right) \\
&\le \frac{M_1(t)}{\mu_i} e^{tM_2(t)} = \frac{M(t)}{\mu_i},
\end{aligned} \tag{23}$$

where $M(t) := M_1(t) e^{tM_2(t)}$ is a positive and continuous function. Since $\lim_{i\to\infty} \mu_i = \infty$, $\lim_{i\to\infty} Y(t) = 0$ follows directly from (23). Furthermore, with (13) and $X_0 = I$, for all $t > 0$,

$$\begin{aligned}
&\|X(t)\| \\
&\le e^{\mu(M_{11}-K_i)t} + \|M_{12}\| \int_0^t \underbrace{e^{\mu(M_{11}-K_i)(t-\tau)}}_{\in(0,1]} \|Y(\tau)\| \, d\tau \\
&\le e^{\mu(M_{11}-K_i)t} + \frac{\|M_{12}\|}{\mu_i} \int_0^t M(\tau) \, d\tau \\
&\le e^{\mu(M_{11}-K_i)t} + \frac{\|M_{12}\|}{\mu_i} t \max_{\tau\in[0,t]} (M(\tau)).
\end{aligned}$$

Therefore, $\lim_{i\to\infty} X(t) = 0$, which completes the proof of Part 1.

PART 2: Substituting (14) into (13) and using the initial conditions $X_0 = 0$ and $Y_0 = I$ yields, after interchange of integration in the second term,

$$\begin{aligned}
X(t) &= \int_0^t e^{(M_{11}-K_i)(t-\tau)} M_{12} e^{M_{22}\tau} \, d\tau \\
&+ \int_0^t \int_s^t e^{(M_{11}-K_i)(t-\tau)} M_{12} e^{M_{22}(\tau-s)} M_{21} X(s) \, d\tau \, ds,
\end{aligned}$$

and, therefore,

$$\begin{aligned}
\|X(t)\| &\le \|M_{12}\| \int_0^t e^{\mu(M_{11}-K_i)(t-\tau)} e^{\|M_{22}\|\tau} \, d\tau \\
&+ \int_0^t \|M_{12}\|\|M_{21}\| \int_s^t e^{\mu(M_{11}-K_i)(t-\tau)} e^{\|M_{22}\|(\tau-s)} \, d\tau \, \|X(s)\| \, ds.
\end{aligned} \tag{24}$$

Now, consider the substitutions $\tau \to t-\tau$ for the first term in (24) and $\tau \to t+s-\tau$ for the inner integral of the second term, which yields

$$\begin{aligned}
\|X(t)\| &\le \|M_{12}\| \int_0^t e^{\|M_{22}\|(t-\tau)} e^{\mu(M_{11}-K_i)\tau} \, d\tau \\
&+ \int_0^t \|M_{12}\|\|M_{21}\| \int_s^t e^{\|M_{22}\|(t-\tau)} e^{\mu(M_{11}-K_i)(\tau-s)} \, d\tau \, \|X(s)\| \, ds.
\end{aligned} \tag{25}$$

Comparing this inequality to (16), we find that we can obtain (25) from (16) by the substitutions $\|Y(\cdot)\| \to \|X(\cdot)\|$, $\|M_{12}\| \to \|M_{21}\|$, and $\|M_{21}\| \to \|M_{12}\|$. Therefore, we can derive an upper bound on $\|X(t)\|$ the same way as in Part 1. Corresponding to (23), for all $t \geq 0$,

$$\|X(t)\| \leq \frac{L(t)}{\mu_i},$$

where we obtain $L(t)$ from $M(t)$ by substituting $\|M_{12}\| \to \|M_{21}\|$ and $\|M_{21}\| \to \|M_{12}\|$. Thus, $\lim_{i\to\infty} X(t) = 0$ and, with (14) and $Y_0 = I$, we have, for all $t > 0$,

$$\begin{aligned}\|Y(t) - e^{M_{22}t}\| &\leq \|M_{21}\| \int_0^t e^{\mu(M_{22})(t-\tau)} \|X(\tau)\|\, d\tau \\ &\leq \frac{\|M_{21}\|}{\mu_i} \int_0^t e^{\mu(M_{22})(t-\tau)} L(\tau)\, d\tau \\ &\leq \frac{\|M_{21}\|}{\mu_i} t \max_{\tau\in[0,t]} \left(e^{\mu(M_{22})(t-\tau)} L(\tau)\right).\end{aligned}$$

Therefore, $\lim_{i\to\infty}\|Y(t) - e^{M_{22}t}\| = 0$ and thus $\lim_{i\to\infty} Y(t) = e^{M_{22}t}$, which completes the proof of Part 2.

PART 3: The matrix exponential

$$\mathcal{X}(t) := \exp\left(\begin{bmatrix} M_{11} - K_i & M_{12} \\ M_{21} & M_{22}\end{bmatrix} t\right)$$

is the unique solution to the linear matrix ODE, [4],

$$\dot{\mathcal{X}}(t) = \begin{bmatrix} M_{11} - K_i & M_{12} \\ M_{21} & M_{22}\end{bmatrix} \mathcal{X}(t),\ t \geq 0,\ \mathcal{X}(0) = I. \quad (26)$$

Note that this implies continuity of $\mathcal{X}(\cdot)$. By subdividing $\mathcal{X}(t)$ into block matrices of appropriate dimensions,

$$\mathcal{X}(t) = \begin{bmatrix} \mathcal{X}_{11}(t) & \mathcal{X}_{12}(t) \\ \mathcal{X}_{21}(t) & \mathcal{X}_{22}(t)\end{bmatrix},$$

we can write (26) equivalently as

$$\begin{bmatrix} \dot{\mathcal{X}}_{11}(t) \\ \dot{\mathcal{X}}_{21}(t)\end{bmatrix} = \begin{bmatrix} M_{11} - K_i & M_{12} \\ M_{21} & M_{22}\end{bmatrix} \begin{bmatrix} \mathcal{X}_{11}(t) \\ \mathcal{X}_{21}(t)\end{bmatrix}, \begin{bmatrix} \mathcal{X}_{11}(0) \\ \mathcal{X}_{21}(0)\end{bmatrix} = \begin{bmatrix} I \\ 0\end{bmatrix}, \quad (27)$$

$$\begin{bmatrix} \dot{\mathcal{X}}_{12}(t) \\ \dot{\mathcal{X}}_{22}(t)\end{bmatrix} = \begin{bmatrix} M_{11} - K_i & M_{12} \\ M_{21} & M_{22}\end{bmatrix} \begin{bmatrix} \mathcal{X}_{12}(t) \\ \mathcal{X}_{22}(t)\end{bmatrix}, \begin{bmatrix} \mathcal{X}_{12}(0) \\ \mathcal{X}_{22}(0)\end{bmatrix} = \begin{bmatrix} 0 \\ I\end{bmatrix}. \quad (28)$$

Note that (27) and (28) represent the matrix ODEs considered in Part 1 and Part 2, respectively. Using the results (9), (10), we therefore conclude

$$\begin{aligned}&\lim_{i\to\infty} \exp\left(\begin{bmatrix} M_{11} - K_i & M_{12} \\ M_{21} & M_{22}\end{bmatrix}\right) \\ &= \lim_{i\to\infty} \begin{bmatrix} \mathcal{X}_{11}(1) & \mathcal{X}_{12}(1) \\ \mathcal{X}_{21}(1) & \mathcal{X}_{22}(1)\end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \exp(M_{22})\end{bmatrix}.\end{aligned}$$

■

## III. TIME SCALE SEPARATION ALGORITHM

In this section, we present an application of Theorem 2.1. We derive a technique for simplifying a description of a control system with inner and outer feedback loops, where the time scale of the inner loops can be assumed to be sufficiently smaller than the time scale of the outer loops. We are interested in deriving a discrete-time model at the slow rate, i.e. that is relevant to the outer loop, while assuming infinitely fast inner loops. The problem is formulated in Sec. III-A and the time scale separation algorithm is derived in Sec. III-B.

### A. Problem Formulation

Consider the continuous-time, linear time-invariant system (time argument $t$ in $x$ and $u$ omitted for convenience)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2\end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22}\end{bmatrix} \begin{bmatrix} x_1 \\ x_2\end{bmatrix} + \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22}\end{bmatrix} \begin{bmatrix} u_1 \\ u_2\end{bmatrix}, \quad (29)$$

where $x_1 \in \mathbb{R}^{n_1}$, $x_2 \in \mathbb{R}^{n_2}$, $u_1 \in \mathbb{R}^{m_1}$, and $u_2 \in \mathbb{R}^{m_2}$. We require $n_1 = m_1$ and $B_{11}$ being invertible. The input $u_2$ changes at a rate $T$; $u_1$ may change at a faster rate. Local proportional feedback is applied on the states $x_1$ through the input $u_1$,

$$u_1 = F(v - x_1), \quad (30)$$

with the gain matrix $F$ and the reference input $v \in \mathbb{R}^{n_1}$ that also changes at rate $T$. The control system is depicted in Fig. 1. We assume that the dynamics of the local feedback

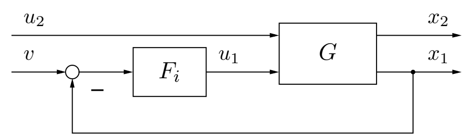


Fig. 1. Control system with local feedback on $x_1$.

loop closed through (30) are sufficiently faster than the dynamics of the remaining states $x_2$.

We are ultimately interested in a discrete-time model of the system (29) with local feedback (30) discretized at sampling rate $T$, that is we are interested in the system from inputs $(v, u_2)$ to outputs $(x_1, x_2)$ as shown in Fig. 1. This model can for example be used to design an outer loop controller.

We will consider the limiting case, where the local feedback loop closed by (30) is made arbitrarily fast, i.e. where the eigenvalues of $A_{11} - B_{11}F$ go to negative infinity. Precisely, we consider a sequence of feedback controllers $F = F_i, i = 1, 2, \ldots, \infty$, where $\mu(-B_{11}F_i) \to -\infty$ as $i \to \infty$. For example, the gain matrices may be chosen as

$$F_i = B_{11}^{-1}\mathrm{diag}(k_{i,1}, k_{i,2}, \ldots, k_{i,n_1}), \quad (31)$$

with $k_{i,j} \geq i$ for all $j = 1, \ldots, n_1$, i.e. individual loops are closed on the states $x_1$. In the following, we will derive the discrete-time system of (29) under feedback (30), (31) as $i \to \infty$.

### B. Algorithm

Substituting (30) in (29) yields

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2\end{bmatrix} = \begin{bmatrix} A_{11} - B_{11}F_i & A_{12} \\ A_{21} - B_{21}F_i & A_{22}\end{bmatrix} \begin{bmatrix} x_1 \\ x_2\end{bmatrix} + \begin{bmatrix} B_{11}F_i & B_{12} \\ B_{21}F_i & B_{22}\end{bmatrix} \begin{bmatrix} v \\ u_2\end{bmatrix}.$$

Since the inputs $v$ and $u_2$ are constant over the sampling period $T$ (assuming zero-order hold sampling), this can be rewritten as

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{v}\\ \dot{u}_2\end{bmatrix} = \begin{bmatrix}A_{11}-B_{11}F_i & A_{12} & B_{11}F_i & B_{12}\\ A_{21}-B_{21}F_i & A_{22} & B_{21}F_i & B_{22}\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ v\\ u_2\end{bmatrix}, \tag{32}$$

for $0 \leq t \leq T$. Denoting the vector of states and inputs by $w := [x_1 \quad x_2 \quad v \quad u_2]^T$ and the matrix by $H$, (32) can be rewritten as $\dot{w}(t) = Hw(t)$.

Now, apply the transformation $S$, $\bar{w} := Sw$, with

$$S := \begin{bmatrix}I & 0 & -I & 0\\ -Z & I & 0 & 0\\ 0 & 0 & I & 0\\ 0 & 0 & 0 & I\end{bmatrix}, \quad S^{-1} = \begin{bmatrix}I & 0 & I & 0\\ Z & I & Z & 0\\ 0 & 0 & I & 0\\ 0 & 0 & 0 & I\end{bmatrix},$$

where $Z := B_{21}B_{11}^{-1}$. The transformed system reads

$$\dot{\bar{w}}(t) = \bar{H}\bar{w}(t), \tag{33}$$

with

$$\bar{H} := SHS^{-1} = \begin{bmatrix}A_{11}+A_{12}Z-B_{11}F_i & \bar{H}_{12} & \bar{H}_{13} & \bar{H}_{14}\\ \bar{H}_{21} & \bar{H}_{22} & \bar{H}_{23} & \bar{H}_{24}\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix},$$

$$\begin{aligned}&\bar{H}_{12} = A_{12}, \quad \bar{H}_{13} = A_{11}+A_{12}Z, \quad \bar{H}_{14} = B_{12},\\ &\bar{H}_{21} = A_{21}-ZA_{11}+(A_{22}-ZA_{12})Z,\\ &\bar{H}_{22} = A_{22}-ZA_{12},\\ &\bar{H}_{23} = A_{21}-ZA_{11}+(A_{22}-ZA_{12})Z,\\ &\bar{H}_{24} = B_{22}-ZB_{12}.\end{aligned}$$

Note that via this transformation only the block $\bar{H}_{11} = A_{11} + A_{12}Z - B_{11}F_i$ depends on the feedback gain matrix $F_i$.

We discretize the system (33) with sampling time $T$,

$$\bar{w}(t+T) = \exp(T\bar{H})\bar{w}(t). \tag{34}$$

Now, consider the limiting case for infinitely fast local feedback (31) as $i \to \infty$. We have $\mu(-B_{11}F_i) = \mu(-B_{11}B_{11}^{-1}\text{diag}(k_{i,1}, k_{i,2}, \ldots, k_{i,n_1})) = \mu(-\text{diag}(k_{i,1}, k_{i,2}, \ldots, k_{i,n_1})) = \max_{j=1,\ldots,n_1}(-k_{i,j}) \to -\infty$ as $i \to \infty$. Therefore, we can apply Theorem 2.1 to the matrix $\exp(T\bar{H})$. We obtain

$$\lim_{i\to\infty} \exp(T\bar{H}) = \begin{bmatrix}0 & 0 & 0 & 0\\ 0 & \hat{H}_{22} & \hat{H}_{23} & \hat{H}_{24}\\ 0 & 0 & I & 0\\ 0 & 0 & 0 & I\end{bmatrix}, \tag{35}$$

where $\hat{H}_{22}$, $\hat{H}_{23}$, and $\hat{H}_{24}$ are defined from

$$\begin{bmatrix}\hat{H}_{22} & \hat{H}_{23} & \hat{H}_{24}\end{bmatrix} = \begin{bmatrix}I & 0 & 0\end{bmatrix}\exp\left(T\begin{bmatrix}\bar{H}_{22} & \bar{H}_{23} & \bar{H}_{24}\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\right).$$

Using the fact that $\exp(TH) = \exp(TS^{-1}\bar{H}S) = S^{-1}\exp(T\bar{H})S$, [4], we obtain the discretization of the system (32), as $i \to \infty$,

$$w(t+T) = \lim_{i\to\infty}\left(\exp(TH)\right)w(t),$$

where

$$\lim_{i\to\infty}\exp(TH) = S^{-1}\lim_{i\to\infty}\left(\exp(T\bar{H})\right)S = \begin{bmatrix}0 & 0 & I & 0\\ -\hat{H}_{22}Z & \hat{H}_{22} & \hat{H}_{23}+Z & \hat{H}_{24}\\ 0 & 0 & I & 0\\ 0 & 0 & 0 & I\end{bmatrix}.$$

This can be rewritten as

$$\begin{aligned}\begin{bmatrix}x_1(t+T)\\ x_2(t+T)\end{bmatrix} &= \begin{bmatrix}0 & 0\\ -\hat{H}_{22}Z & \hat{H}_{22}\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}\\ &\quad + \begin{bmatrix}I & 0\\ \hat{H}_{23}+Z & \hat{H}_{24}\end{bmatrix}\begin{bmatrix}v(t)\\ u_2(t)\end{bmatrix},\end{aligned} \tag{36}$$

which is the desired discrete-time system of (29) under feedback (30), (31) as $i \to \infty$. Note, in particular, that $x_1(t+T) = v(t)$ as expected, i.e. we have ideal reference tracking. We will refer to $x_1$ as the *residualized states*. The simplification of the matrix exponential in Theorem 2.1 for $i \to \infty$ corresponds to the simplified dynamics for the residualized states due to high gain inner loop feedback.

## IV. Application to Multi-Loop Control of an Inverted Pendulum

In this section, the time scale separation algorithm presented in Sec. III is applied to the problem of designing a two-loop cascaded control system for stabilizing an inverted pendulum. The system and the control architecture are described in Sec. IV-A. In Sec. IV-B, the time scale separation technique is applied to derive a simplified model. With this model, a state feedback controller is designed. Experimental results of the closed-loop operation of the pendulum are given in Sec. IV-C.

### A. System description

The inverted pendulum (see Fig. 2) is pivoted at the ground; it has one rotational degree of freedom. On the pendulum, a second body (referred to as the *module*) is mounted. A DC motor on the module can rotate the module with respect to the pendulum via a bevel gear. The DC motor features a built-in velocity controller. Two physical effects are utilized to stabilize the inverted pendulum: First, torque is exerted on the pendulum when the module is actuated, and, second, the center of mass of the overall system is shifted by moving the module.

Two encoders on the module and at the pendulum pivot are used to measure the angles $\phi$ and $\psi$ in radians. Rate gyros mounted on the pendulum measure the pendulum angular velocity $\dot{\phi}$ and an encoder on the motor is used to determine the motor velocity, which is proportional to the module velocity $\dot{\psi}$.

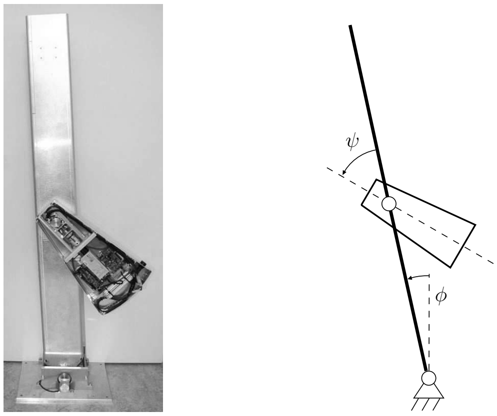


Fig. 2. The inverted pendulum. Left: Photo of the real system. Right: Schematic drawing with module angle $\psi$ and pendulum angle $\phi$.

A linear state space model of the system, which was derived from first principles modeling, is given by

$$\dot{x} = Ax + Bu_1, \tag{37}$$

$$A = \begin{bmatrix} 0 & 0 & -15.78 & -8.04 \\ 0 & 0 & -2.24 & 11.58 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 2.52 \\ 0.14 \\ 0 \\ 0 \end{bmatrix},$$

where the states are $x = [\dot{\psi}, \dot{\phi}, \psi, \phi]^T$ and $u_1$ is the torque at the module. The open loop poles are at $\pm 4.05\, i\,\mathrm{rad/s}$ and $\pm 3.5\,\mathrm{rad/s}$.

Our objective is to design a controller for stabilizing the system about the equilibrium of an upright pendulum ($\phi = 0$) and a downward module ($\psi = 0$). We use a two-loop control architecture: a fast inner feedback loop operating at 1 kHz that tracks commanded velocities and a slower outer loop at 100 Hz generating the velocity commands from state measurements, see Fig. 3. For the inner loop controller $C_v$, we employ the local velocity controller on the DC motor.

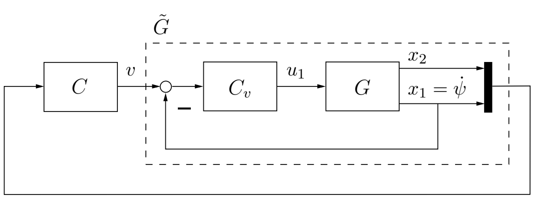


Fig. 3. The control architecture for balancing the inverted pendulum. The block $G$ represents the pendulum, $C_v$ denotes the motor velocity controller, $\tilde{G}$ combines these two blocks, and $C$ is the outer controller.

In model (37), we assume that we can control the torque at the module directly. In reality, however, we can only control the torque at the motor, which is translated to the torque at the module in a nontrivial way through a transmission system, which involves nonlinearities like kinetic and static friction and backlash. The approach we take avoids modeling these nonidealities. By closing the inner loop, we can take the abstract view of the motor and its controller as a system that achieves a commanded module velocity sufficiently fast. In fact, we consider the ideal case of an infinitely fast inner loop control system: we apply the time scale separation algorithm from Sec. III in order to obtain a simplified model of the local feedback system $\tilde{G}$. With this model, we can design the outer loop controller $C$ without detailed knowledge of the controller $C_v$ and the nonidealities involved in the inner loop. Note, however, that the effect of the inner loop itself is taken into account in the simplified model by the time scale separation algorithm.

The assumption that the motor controller tracks the reference velocity sufficiently fast translates to the requirement that the inner closed-loop system $\tilde{G}$ (from $v$ to $\dot{\psi}$) operates at a sufficiently smaller time scale than the outer closed-loop system. In Fig. 4, the response of the system $\tilde{G}$ to a step change in the reference velocity is shown. The time constant of the motor response is roughly 0.03 s, which is about one order of magnitude smaller than the time constants of the physical system (37). Therefore, the assumption made

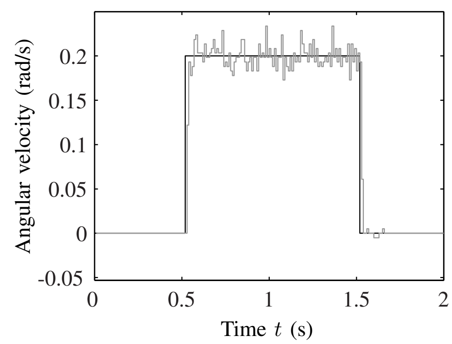


Fig. 4. Inner loop subsystem $\tilde{G}$: Response of the module velocity $\dot{\psi}$ (gray) to changes in the reference velocity $v$ (black).

in Sec. III-A that the local feedback dynamics are sufficiently faster than the dynamics of the remaining states is valid. It is thus legitimate to approximate the motor controller $C_v$ by a controller with an infinite gain of the form (30), (31).

### B. Controller design using time scale separation algorithm

We approximate the inner closed-loop system $\tilde{G}$ using the time scale separation algorithm presented in Sec. III. Applying (36) on the system (37) with sampling time $T = 0.01$ yields

$$\begin{bmatrix} x_1(t+T) \\ x_2(t+T) \end{bmatrix} = A_d \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + B_d v(t), \tag{38}$$

$$A_d = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -0.054 & 1.001 & -0.014 & 0.12 \\ 0 & 0 & 1.0 & 0 \\ -5.4e{-}4 & 0.010 & -7.0e{-}5 & 1.001 \end{bmatrix}, \; B_d = \begin{bmatrix} 1.0 \\ 0.054 \\ 0.010 \\ 5.4e{-}4 \end{bmatrix},$$

where $x_1 = \dot{\psi}$ is the residualized state, $x_2$ includes all other states, and the desired module velocity $v(t)$ is the new input. Note in particular that by the high gain assumption the commanded module velocity is achieved in one time step, $x_1(t) = v(t - T)$, irrespective of all other states.

The linear discrete-time model (38) is used to design an infinite horizon LQR controller. In addition to weights on states and control input $v(t)$, we also penalize the difference in controller commands $v(t)-v(t-T)$. This is equivalent to penalizing torque, which is the input to the original system (37). Thus, we have the cost function

$$J(v) = \sum_{k=0}^{\infty} (x_2(kT))^T Q x_2(kT) + r(v(kT))^2 + \xi\big(v(kT) - v((k-1)T)\big)^2, \tag{39}$$

with weighting matrix $Q$ and scalars $r, \xi$. Since $v((k–1)T) = x_1(kT)$, we can reformulate (39) as an LQR design problem with nonzero weights on state and input cross terms

$$J(v) = \sum_{k=0}^{\infty} (x(kT))^T \begin{bmatrix} \xi & 0 \\ 0 & Q \end{bmatrix} x(kT) + (r+\xi)\big(v(kT)\big)^2 + 2(x(kT))^T \begin{bmatrix} -\xi \\ 0 \end{bmatrix} v(kT), \tag{40}$$

which can be solved using standard LQR design tools, [8]. A stabilizing feedback gain vector that results from suitable weights is

$$l = \begin{bmatrix} l_1 & l_2 & l_3 & l_4 \end{bmatrix} = \begin{bmatrix} -0.23 & -9.99 & 3.66 & -34.6 \end{bmatrix}.$$

The closed-loop poles are at 0.835, 0.966, 0.966, and 0.978. As expected, there is one pole corresponding to the residualized state that is considerably faster than the other poles.

Using the approximation $x_1(t) = v(t-T)$, we implement the following control law:

$$v(t) = -l_1 v(t-T) - \begin{bmatrix} l_2 & l_3 & l_4 \end{bmatrix} x_2(t), \tag{41}$$

where the states $x_2$ are measured as described in Sec. IV-A. Note that measurements of the residualized state are not required for controlling the system.

### C. Experimental results

The pendulum was operated with the controller architecture shown in Fig. 3 and with the control law (41) implemented for $C$. Typical state measurements during balancing of the pendulum are shown in Fig. 5, where at time $t \approx 3\,\text{s}$ the system was disturbed by an impact on the pendulum. From Fig. 6, we can see that the assumption of fast velocity command tracking is valid.

## V. Concluding Remarks

The time scale separation algorithm for a multi-loop control system that we presented in Sec. III is one possible application of the matrix exponential result in Theorem 2.1. However, the limiting property of the matrix exponential might also be beneficial for understanding other problems involving fast and slow dynamics.

Similar results to the presented time scale separation algorithm might also be obtained using singular perturbation methods (see e.g. [9]). A detailed comparison of these techniques is beyond the scope of this paper, but shall be included in a later publication on this topic.

Furthermore, we are working on possible relaxations of the assumptions on the system (29) (especially the input matrix $B_{11}$ being invertible) and on extending the time scale separation technique to general nonlinear controllers that are sufficiently fast.

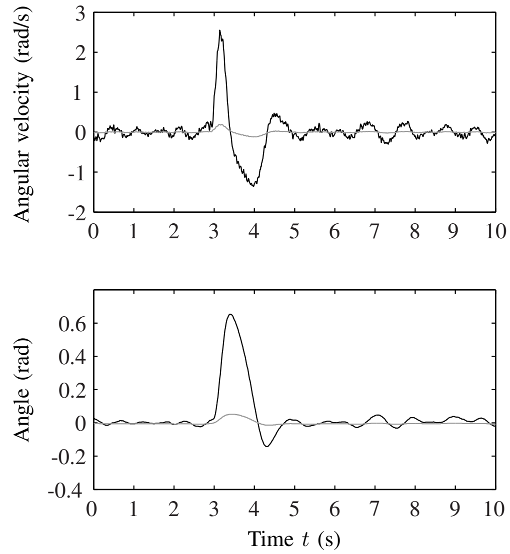


Fig. 5. State measurements of the inverted pendulum in closed-loop operation. Top: Angular velocity of the module $\dot\psi$ (black) and the pendulum $\dot\phi$ (gray). Bottom: Angles of the module $\psi$ (black) and the pendulum $\phi$ (gray). At $t \approx 3$s the system was disturbed by an impact on the pendulum.

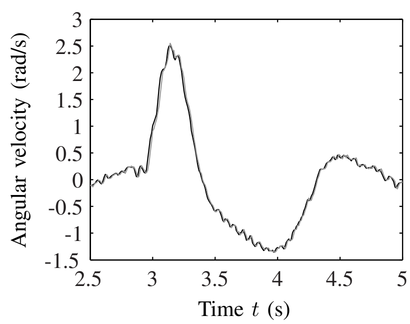


Fig. 6. Fast velocity tracking: Commanded velocity $v$ (black) and actual measured module velocity $\dot\psi$ (gray).

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